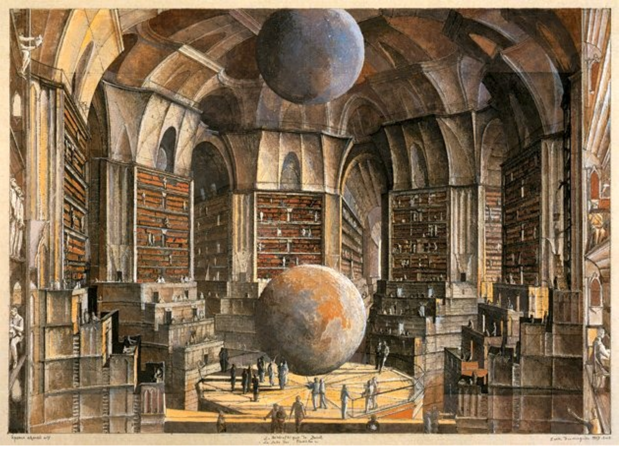
December 2025, No 14

Scripta-Ingenia

Open Letter to Future Generations



in Meet the Library of Babel: Every Possible Combination of Letters That Has Been*, by Jolene Creighton, Futurism (Sep 29, 2015).*

The world is changing, faster and faster. I've always tried to think like a visionary, to look at least ten years ahead. I was born in the seventies of the twentieth century, by which time most of Jules Verne's prophecies had already been fulfilled. Today, on the Winter Solstice of 2025, decades after Mount St. Helens (1980) and nearly two millennia since Vesuvius (Aug 24, A.D. 79), I find myself reflecting on a shift in perspective: a few years ago, I realized that looking just ten years ahead was no longer enough. I began planning in twenty-year spans. And now, unsettlingly, even that feels insufficient.

We stand at a turning point. The choices we make today, regarding what we build, release, ignore, or amplify, will define the world you inherit. If the future brings wonder or ruin, it will not be the fault of machines. It will be ours.

Artificial intelligence, unlike us, does not wrestle with good and evil. To it, morality is as neutral as color. Black and white are distinguishable, but neither is preferred. And so, the burden of meaning, and of consequence, remains firmly human.

The dawn of the AI age has come not with a bang, but with the whisper of a billion pages. Every second gives rise to volumes: posts, data points, code snippets, synthetic essays; flooding into a library that rivals Borges's invention. The Library of Babel, a labyrinth that contains every possible book. Most are unreadable. Many are indistinguishable from truth.

All truths, therefore none. To future readers and thinkers, I leave a humble warning: Intelligence, artificial or human, is not wisdom. In the infinite archive, every truth lies buried beneath countless variations of itself. Algorithms may lead you to a page, but not to meaning.

And herein lies the paradox. In a world where every answer already exists, how do you find the right question? When every contradiction lives alongside its negation, we begin to drown not in lies, but in an ocean of everything. Over-information, not misinformation, may prove the quieter catastrophe.

Just as real numbers once promised to capture the physical world with perfect precision, today's endless flow of digital content offers the illusion of certainty. But just as the discovery of irrationality and the invention of imaginary quantities exposed the limits of numerical representation, showing that not all quantities can be finitely expressed or directly apprehended, we must learn that not all knowledge can be indexed, accessed, or even understood. Digital abundance is not clarity; it is complexity made visible.

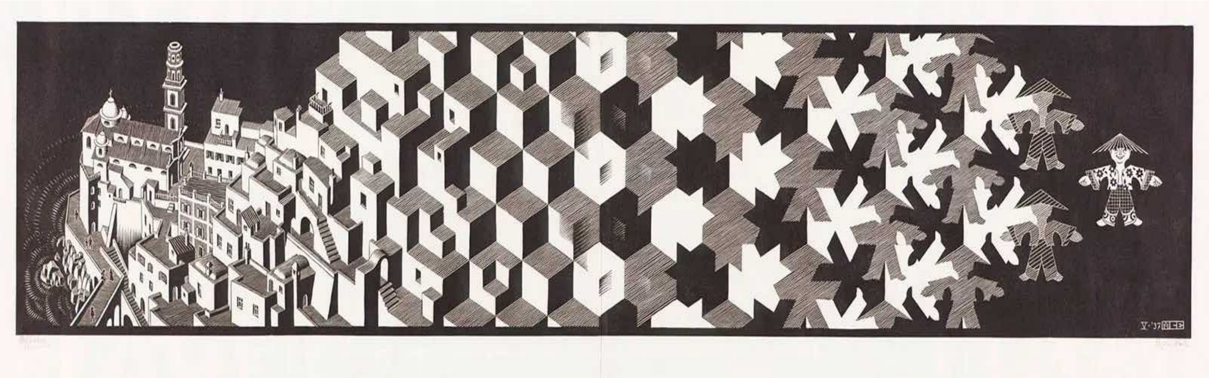
Like explorers in a fractal maze, we must learn to navigate not merely through data, but through doubt. To discern signal from echo. To treasure silence as much as accumulation. Let AI be a lantern, not a map.

In your era, machines may compose symphonies and philosophies alike. But the task remains yours: to choose, to question, to discard. In Borges's library, every book exists. But meaning begins only when you decide which shelves to leave behind.

Português
*A **Scripta-Ingenia** assume-se como uma revista de divulgação científica dedicada a temas da ciência e da tecnologia, abrangendo todas as áreas do saber no domínio das ciências exatas e aplicadas. Tem igualmente interesse por artigos de opinião — científicos ou não — desde que escritos por autores da área das ciências e da engenharia, refletindo as suas perspetivas enquanto membros dessa comunidade académica e profissional. Este é o seu número 14, correspondente ao Solstício de Inverno de 2025.*

English
***Scripta-Ingenia** presents itself as a journal for scientific dissemination, addressing topics in science and technology across all fields of knowledge within the domain of exact and applied sciences. It also welcomes opinion articles — scientific or otherwise — provided they are written by authors in the fields of science and engineering, and express their perspectives as members of that academic and professional community. This is issue number 14, corresponding to the Winter Solstice of 2025.*

Director and Chief Editor — Nelson Martins-Ferreira
GTLab-CDRSP-ESTG, Polytechnic of Leiria



M.C. Escher, Metamorphosis I, woodcut, printed on two sheets, May 1937

We do not know space. We do not see it, we do not hear it, we do not feel it. We are standing in the middle of it, we ourselves are part of it, but we know nothing about it. — M.C. Escher

Ficha Técnica
Proprietário: Politécnico de Leiria - NIPC 506971244
Morada e Sede de Redação: Rua General Norton de Matos; Apartado 4133; 2411-901 Leiria, Portugal
Número de registo: 127780
Diretor e Editor: Nelson Martins-Ferreira
Contacto: martins.ferreira@ipleiria.pt
Online: `https://cdrsp.ipleiria.pt/scriptaingenia`
Estatuto editorial: (a) a Scripta-Ingenia é uma revista de cariz científico e académico publicada pelo Politécnico de Leiria; (b) tem periodicidade anual, podendo excecionalmente publicar números especiais, ora libertados nos equinócios, ora nos solstícios; (c) as principais línguas são o inglês e o português, podendo contudo publicar artigos noutras línguas; (d) apresenta-se como veículo de divulgação da investigação e desenvolvimento em áreas de cariz técnico-científico, ou outras afins, tais como todas as disciplinas clássicas do conhecimento humano.

Integração Agrivoltáico na Produção de Maçã Fuji em Portugal

by M. Gameiro(1), R. Manso(2), N. Martins-Ferreira(1,2), N. Monteiro(2)

(1) Centre for Rapid and Sustainable Product Development
(2) School of Technology and Management
Polytechnic of Leiria
Author email: nuno.f.monteiro@ipleiria.pt

Acknowledgment: Este trabalho obteve apoio financeiro total ou parcial da Agenda de Investigação e Inovação para a Sustentabilidade da Agricultura, Alimentação e Agroindústria (Aviso n.º 09/C05-i03/2021), no âmbito do projeto PRR-C05-i03-I-000251 (FruitPV), ao abrigo do Plano de Recuperação e Resiliência (PRR) e Fundos Europeus NextGeneration EU.

Abstract Agrivoltaics (AgriPV) represents an emerging approach that integrates agricultural production and photovoltaic energy generation within the same space, thereby enhancing land-use efficiency and strengthening the sustainability of production systems. In fruit cultivation, international trials have consistently demonstrated benefits such as reduced sunburn in apple orchards in France, improved water-use efficiency in vineyards and citrus orchards in Japan, enhanced fruit quality and heatwave mitigation in Spain and Italy, and water conservation and microclimatic stabilization in orchards across the United States. In Portugal, the FruitPV Project, developed at INIAV's Innovation Hub in Alcobaça, constitutes the first structured implementation of this technology in Fuji apple production. The experimental orchard—comprising 1,269 trees across 0.4 ha—integrates five pilot AgriPV system models (100 kWp; 150 MWh/year) to simultaneously assess energy production, plant physiology, and fruit quality. The 2025 update includes results on fruit set rate, fruit growth, and net photosynthesis, highlighting superior performance in systems with semi-transparent panels and trees trained under the Guyot system. Preliminary findings demonstrate that AgriPV integration enables the coexistence of Fuji apple production and solar energy generation without compromising agronomic performance, positioning FruitPV as a replicable model with high potential for broader application in other regions and fruit crops.

Keywords: Agrivoltaics, Fuji apple, photovoltaic systems, sustainable agriculture, microclimate regulation

Resumo O agrivoltaico (AgriPV) constitui uma abordagem emergente que integra produção agrícola e geração fotovoltaica no mesmo espaço, aumentando a eficiência do uso do solo e reforçando a sustentabilidade dos sistemas produtivos. Na fruticultura, os ensaios internacionais indicam benefícios consistentes: redução de escaldão em macieiras em França, melhoria da eficiência hídrica em vinhas e pomares de citrinos no Japão, maior qualidade do fruto e mitigação de ondas de calor em Espanha e Itália, e conservação de água e estabilização microclimática em pomares dos Estados Unidos. Em Portugal, o Projeto FruitPV, desenvolvido no Polo de Inovação do INIAV em Alcobaça, representa a primeira implementação estruturada desta tecnologia na produção de maçã Fuji. O pomar experimental, com 1269 árvores em 0,4 ha, integra cinco modelos piloto de sistemas AgriPV (100 kWp; 150 MWh/ano), avaliando simultaneamente produção energética, fisiologia vegetal e qualidade dos frutos. A atualização de 2025 incorpora resultados relativos à taxa de vingamento, crescimento dos frutos e fotossíntese líquida, evidenciando melhor desempenho das modalidades com painéis semi-transparentes e das árvores conduzidas em sistema Guyot. Os resultados preliminares demonstram que a integração AgriPV permite compatibilizar a produção de maçã Fuji com geração de energia solar sem comprometer o desempenho agronómico, posicionando o FruitPV como um modelo com elevado potencial de replicação noutras regiões e culturas frutícolas.

Palavras-chave: Agrivoltaico, fruticultura, eficiência hídrica, sustentabilidade ambiental, resiliência agrícola, inovação tecnológica, projeto FruitPV.

1 Introdução

O aquecimento global é uma das maiores ameaças atuais ao ambiente e à sociedade. Como o setor energético responde por cerca de 75% das emissões globais de dióxido de carbono, a transição acelerada para fontes renováveis torna-se imperativa [1,2]. A energia solar fotovoltaica (FV) tem registado avanços notáveis nas últimas décadas [3]; contudo, a elevada necessidade de área — cerca de 2,0 ha por MW [4] — intensifica a competição pelo uso do solo, sobretudo com a agricultura. Paralelamente, a agricultura enfrenta desafios crescentes ligados às alterações climáticas, à escassez de água e à dependência de combustíveis fósseis. Neste cenário, os sistemas agrivoltaicos (AgriPV) (figura 2) apresentam-se como uma solução inovadora, permitindo a produção conjunta de alimentos e eletricidade no mesmo terreno. Ao explorar o facto de as plantas utilizarem apenas parte do espectro solar, estes sistemas aumentam a eficiência do uso do solo e oferecem benefícios adicionais, como a redução do consumo hídrico e a proteção das culturas face a fenómenos climáticos extremos [5,6].

Figura 2 Sistema agrivoltáico

Ensaios internacionais confirmam essas vantagens: menor incidência de escaldão em macieiras em França, maior eficiência hídrica em vinhas e pomares de citrinos no Japão, melhoria da qualidade de frutos em Espanha e Itália e conservação da água em pomares nos Estados Unidos. Além dos ganhos produtivos e ambientais, o AgriPV contribui para diversos Objetivos de Desenvolvimento Sustentável (ODS), incluindo a erradicação da pobreza (ODS 1), a segurança alimentar (ODS 2), o acesso a energia limpa (ODS 7) e a mitigação das alterações climáticas (ODS 13). Também promove a diversificação económica rural (ODS 8 e 11), práticas agrícolas sustentáveis (ODS 12 e 15) e oportunidades de inovação (ODS 9). Em Portugal, o Projeto FruitPV constitui a primeira experiência piloto de aplicação da tecnologia em fruticultura, avaliando a sua viabilidade técnica, agronómica, energética e ambiental na produção de maçã Fuji. O estudo centra-se na análise de diferentes modelos aplicados a culturas frutícolas e hortícolas, sobretudo nas regiões das Beiras, Serra da Estrela e Oeste. A abordagem combina a instalação e monitorização em pomares modernos de macieiras com simulações digitais e tecnologias de sensorização, de modo a otimizar processos, robustez e eficiência operacional, além de avaliar a adaptação a distintas espécies e condições edafoclimáticas. Ao articular inovação tecnológica com sustentabilidade agrícola, o FruitPV procura gerar especificações técnicas, socioeconómicas, legais e ambientais que orientem a implementação de Comunidades de Energias Renováveis (CERs) e projetos de Autoconsumo Coletivo (AC). O modelo inclui ainda projeções quantificadas de desempenho em termos de produção energética, fisiologia vegetal e qualidade dos frutos. Assim, o FruitPV configura uma abordagem estruturante para reduzir a dependência de combustíveis fósseis no setor agroindustrial e reforçar a aceitação social da tecnologia agroenergética, envolvendo ativamente agricultores, associações, autoridades públicas e empresas do setor.

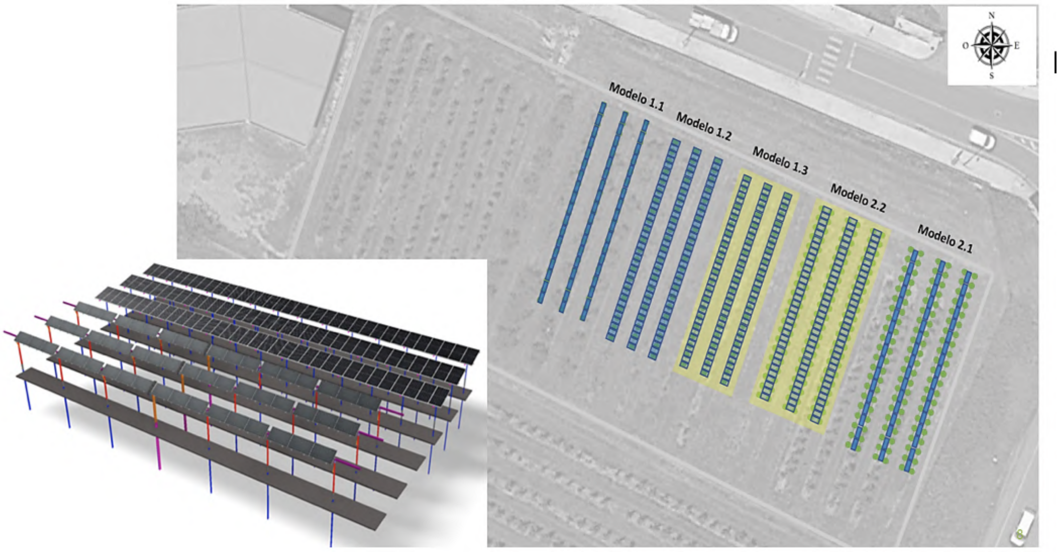


Figura 3: Arquitetura do FruitPV _ Demonstração de Implementação

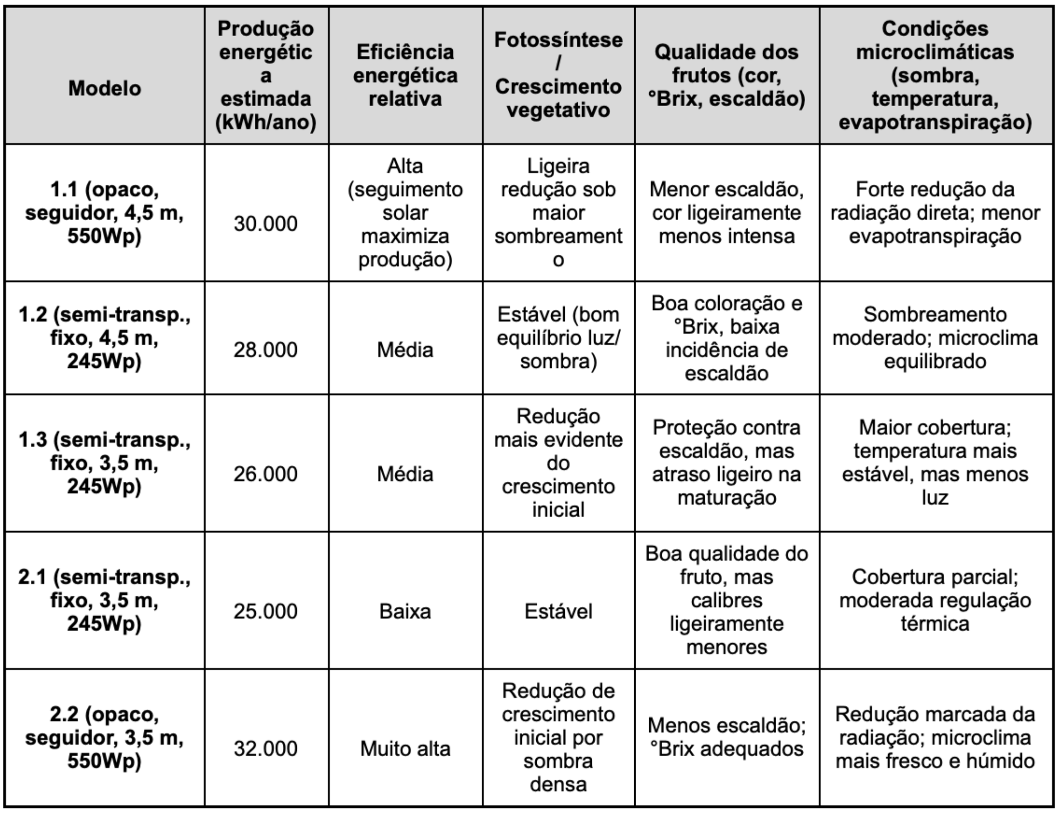
2 Materiais e Métodos

O Projeto FruitPV – Green & Smart Energy Orchards foi implementado em 2024 no Polo de Inovação do Instituto Nacional de Investigação Agrária e Veterinária (INIAV), em Alcobaça, Portugal. A área experimental ocupa 0,4 hectares, onde foram instaladas 1269 macieiras da cultivar Fuji (clone San) em regime de alta densidade, complementadas com árvores da variedade Granny Smith para polinização. As árvores foram conduzidas em dois sistemas distintos: eixo central (linhas simples) e Guyot bidimensional (2D) em linhas duplas. O pomar dispõe de sistema de rega gota-a-gota, assegurando uma gestão eficiente da água. A componente fotovoltaica é constituída por cinco modelos piloto de integração agrivoltaica (figura 3), que variam no tipo de painel (bifacial opaco ou semitransparentes, com 40% de transparência), na estrutura de suporte (fixa ou seguidora solar), na altura das armações (entre 3,5 e 4,5 m) e na percentagem de cobertura do solo (34% a 66%). A área total ocupada pelos painéis é de aproximadamente 650 m², correspondendo a uma potência instalada de 100 kWp, com produção anual estimada de 150 MWh, destinada ao autoconsumo do polo experimental.

avaliação do desempenho do sistema, foram instalados sensores de monitorização contínua que recolhem dados relativos a diferentes componentes:

1. Agronómica: crescimento vegetativo, taxa de fotossíntese, transpiração, índice de área foliar, produtividade por árvore e parâmetros de qualidade dos frutos (calibre, ºBrix, coloração e incidência de escaldão solar).
2. Microclimática: temperatura do ar e do solo, humidade relativa, radiação solar incidente e transmitida, e evapotranspiração.
3. Energética: produção elétrica em kWh, eficiência dos painéis, comparação entre modelos fixos e seguidores.

Os dados são recolhidos de forma contínua e organizados em séries temporais para análise comparativa entre os cinco modelos instalados. A análise estatística inclui métodos descritivos e inferenciais, nomeadamente análise de variância (ANOVA), de modo a identificar diferenças significativas entre tratamentos. O acompanhamento decorrerá durante todo o ciclo produtivo, entre 2024 e 2025, permitindo integrar a componente energética com os resultados agronómicos e ambientais.



Modelo	Produção energética estimada (kWh/ano)	Eficiência energética relativa	Fotossíntese / Crescimento vegetativo	Qualidade dos frutos (cor, °Brix, escaldão)	Condições microclimáticas (sombra, temperatura, evapotranspiração)
1.1 (opaco, seguidor, 4,5 m, 550Wp)	30.000	Alta (seguimento solar maximiza produção)	Ligeira redução sob maior sombreamento	Menor escaldão, cor ligeiramente menos intensa	Forte redução da radiação direta; menor evapotranspiração
1.2 (semi-transp., fixo, 4,5 m, 245Wp)	28.000	Média	Estável (bom equilíbrio luz/ sombra)	Boa coloração e °Brix, baixa incidência de escaldão	Sombreamento moderado; microclima equilibrado
1.3 (semi-transp., fixo, 3,5 m, 245Wp)	26.000	Média	Redução mais evidente do crescimento inicial	Proteção contra escaldão, mas atraso ligeiro na maturação	Maior cobertura; temperatura mais estável, mas menos luz
2.1 (semi-transp., fixo, 3,5 m, 245Wp)	25.000	Baixa	Estável	Boa qualidade do fruto, mas calibres ligeiramente menores	Cobertura parcial; moderada regulação térmica
2.2 (opaco, seguidor, 3,5 m, 550Wp)	32.000	Muito alta	Redução de crescimento inicial por sombra densa	Menos escaldão; °Brix adequados	Redução marcada da radiação; microclima mais fresco e húmido

Tabela 1: Comparação dos resultados esperados de produtividade agrícola e energética

2.1 Comparabilidade dos resultados dos modelos pilotos

Neste estudo são avaliados os cinco modelos principais: (i) o sistema AgriPV com módulo monocristalino bifacial opaco, estructura seguidora, instalado de 3,5 a 4,5 m de altura, (ii e iii) o sistema AgriPV com módulo monocristalino bifacial 40% transparente, estrutura fixa, instalados a 4,5 metros de altura, (iv) o sistema AgriPV com módulo monocristalino bifacial 40% transparente, estrutura fixa, instalados a 3,5 metros de altura e (v) o sistema AgriPV com módulo monocristalino bifacial 40% transparente, estrutura seguidora, instalados a 3,5 metros. Em cada modelo, obtêm-se quantidades distintas de energia elétrica e de produção agrícola. A produção de eletricidade é estimada com base em simulações digitais e dados de sensorização local instalados nos pomares experimentais, enquanto a produção agrícola considera indicadores de fisiologia vegetal (fixação de carbono, fluorescência da clorofila a, Non Photochemical Quenching, eficiência no uso da água e da luz) e qualidade da fruta (percentagem de fruta comercializável, coloração, taxa de crescimento e grau Brix), monitorizados no período experimental (2023–2025).

A visão geral destes modelos é apresentada na Tabela 1, que compara os resultados esperados de produtividade agrícola e energética em pomares de macieiras Fuji de Alcobaça sob diferentes arquiteturas de AgriPV.

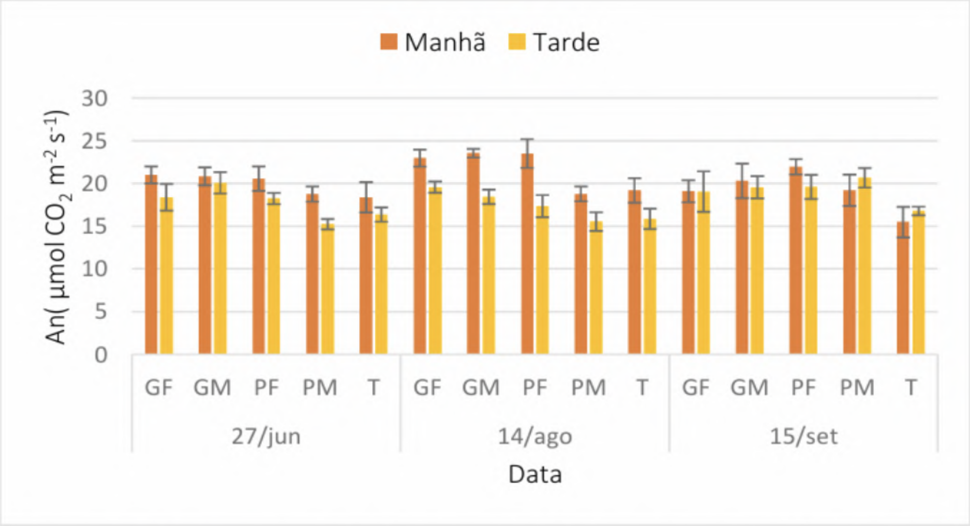


Figura 4 Fotossíntese líquida (An) obtida nas árvores do projeto Fruit-PV GREEN&SMART

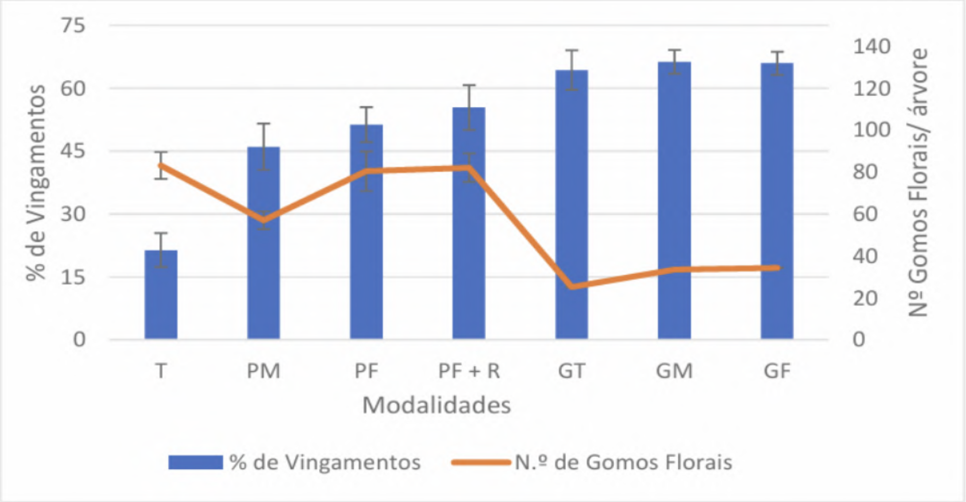


Figura 5 Taxa de vingamento dos frutos do projeto Fruit-PV GREEN&SMART

2.2 Definição de objetivo e âmbito

O presente estudo tem como objetivo avaliar a viabilidade técnica, agronómica e socioeconómica da integração de sistemas agrivoltaicos em pomares de macieiras da variedade *Fuji*, localizados na região de Alcobaça (Portugal), no âmbito do projeto *FruitPV — Green & Smart Energy Orchards* (PRR-C05-i03-I-000251). A análise incide sobre diferentes cenários de implementação, comparando soluções AgroPV e cultivo convencional.

O âmbito do projeto estrutura-se em três dimensões complementares:

- **Agrícola** — monitorização da produtividade e da qualidade dos frutos (% de fruta comercializável, coloração, taxa de crescimento e grau Brix), bem como da fisiologia das plantas (fixação de carbono, fluorescência da clorofila *a*, *Non Photochemical Quenching*, eficiência no uso da água e da luz).
- **Energética** — quantificação da produção elétrica, análise da eficiência e robustez de diferentes configurações fotovoltaicas e avaliação da sua aplicabilidade em autoconsumo individual, autoconsumo coletivo (ACC) e comunidades de energia renovável (CER).
- **Socioeconómica e ambiental** — elaboração de um caderno de especificações que integre requisitos legais, tecnológicos, ambientais e de aceitação social, com vista à replicação do modelo em outras culturas frutícolas e hortícolas conduzidas em linha, em diferentes condições edafoclimáticas.

Deste modo, o projeto visa demonstrar o potencial do agrivoltaico para compatibilizar a produção alimentar e energética no mesmo espaço agrícola, reforçando a resiliência climática, reduzindo os custos energéticos e promovendo a sustentabilidade do setor agroalimentar nacional.

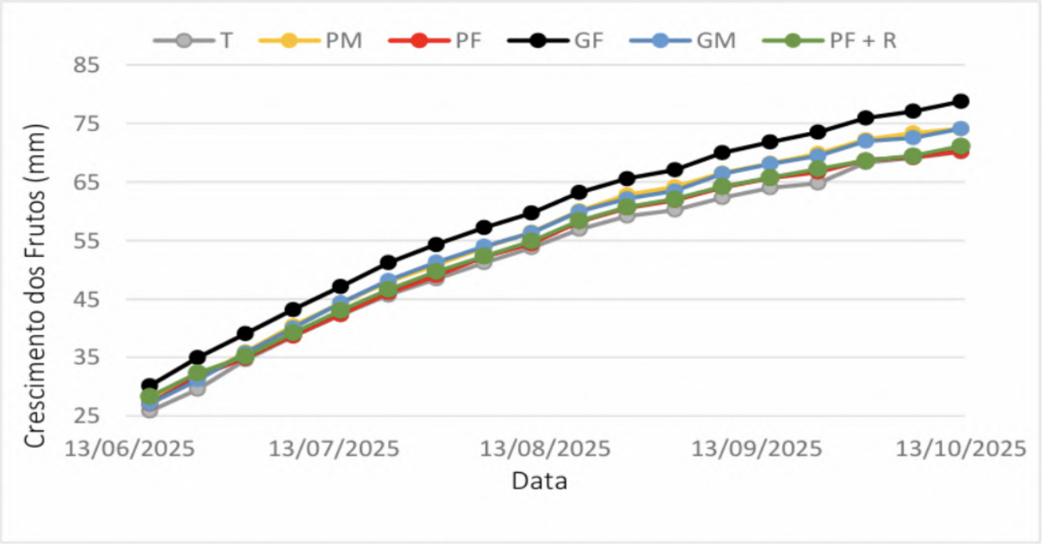


Figura 6 Taxa de crescimento dos frutos do projeto Fruit-PV GREEN&SMART

3 Resultados

3.1 Fotossíntese líquida

A fotossíntese líquida (An) foi quantificada em três momentos representativos do ciclo vegetativo, abrangendo o início do verão, o período de maior *stress* térmico e o final da estação. A Figura 4 apresenta os valores médios de An registados nas diferentes modalidades agrivoltaicas, medidos de manhã e à tarde, permitindo avaliar o efeito do tipo de cobertura fotovoltaica e do sistema de condução na atividade fotossintética das macieiras.

3.2 Taxa de vingamento e gomos florais

A Figura 5 mostra diferenças claras entre modalidades. Verifica-se que as modalidades em sistema Guyot apresentam melhores resultados face às plantações em eixo central; a nível fotovoltaico, são registados melhores resultados nas culturas sombreadas por painéis semitransparentes, comparativamente com as modalidades com painéis móveis.

3.3 Crescimento dos frutos

O crescimento dos frutos foi monitorizado semanalmente através da medição do diâmetro equatorial, utilizando um paquímetro digital, em dez árvores por modalidade, entre 16 de junho e a colheita. Na Figura 6, os resultados indicam que as árvores conduzidas em sistema Guyot apresentaram maior crescimento dos frutos, efeito atribuído tanto à arquitetura bidimensional do sistema como à menor carga frutífera após a monda, que aumentou a disponibilidade de recursos por fruto.

4 Discussão

Os resultados preliminares do *FruitPV* reforçam as conclusões de estudos internacionais, indicando que o agrivoltaico pode simultaneamente proteger as culturas e gerar energia limpa. A integração de diferentes modelos permite comparar soluções e identificar combinações mais adequadas às condições locais.

O sistema bidimensional (2D) em Guyot, associado a painéis semi-transparentes, mostra-se promissor para equilibrar a produção agrícola e energética. A monitorização contínua permitirá validar, em larga escala, os efeitos na produtividade e na qualidade da maçã *Fuji*.

Além disso, o projeto contempla o desenvolvimento de algoritmos de gestão inteligente que poderão otimizar a inclinação dos painéis em função das necessidades da cultura, abrindo caminho para soluções de agricultura de precisão com energia integrada.

5 Conclusão

O Projeto *FruitPV* demonstra que a integração de sistemas agrivoltaicos na fruticultura é viável e vantajosa. A experiência piloto em Alcobaça confirma que é possível conciliar a produção de maçã *Fuji* com a geração de eletricidade solar, obtendo benefícios agronómicos, energéticos e ambientais.

A médio prazo, os resultados deverão apoiar a replicação em pomares comerciais e inspirar a adoção do agrivoltaico em outras culturas, como a vinha e as hortícolas. Este modelo apresenta forte potencial para contribuir para a sustentabilidade agrícola, a resiliência climática e a transição energética em Portugal.

5.1 Limitações

O projeto *FruitPV* apresenta algumas limitações inerentes à sua natureza experimental. A escala piloto e o curto horizonte temporal (2023–2025) restringem a extrapolação dos resultados, sobretudo no que respeita à variabilidade interanual e à adaptação a diferentes condições edafoclimáticas. A ausência de séries históricas nacionais sobre sistemas agrivoltaicos limita ainda a avaliação de impactos de longo prazo no solo, nas plantas e na durabilidade dos painéis. Adicionalmente, a integração das estruturas fotovoltaicas pode condicionar operações agrícolas mecanizadas e requerer soluções técnicas específicas.

5.2 Perspetivas futuras

Para investigações futuras, recomenda-se a expansão geográfica e cultural dos ensaios para outras espécies frutícolas e regiões, a monitorização de longo prazo do desempenho agrícola e energético, e o desenvolvimento de modelos preditivos integrados que associem variáveis climáticas, fisiológicas e elétricas. Estudos complementares de análise de ciclo de vida (*Life Cycle Assessment* – LCA) e de aceitação socioeconómica poderão consolidar a avaliação global da sustentabilidade e viabilidade de adoção do modelo AgroPV em larga escala.

Bibliografia

[1] López, G., Pasquali, A., Hitte, V., Lesniak, V., Bregeon, M., Persello, S., Juillion, P., Chopard, J., & Fumey, D. (2025). Sun Protection for Fruit: Dynamic Agrivoltaics Reduces Apple Temperature and Sunburn Damage. *AgriVoltaics Conference Proceedings*, 3. https://doi.org/10.52825/agripv.v3i.1371

[2] Juillion, P., Lopez, G., Verambre, G., Génard, M., Lesniak, V., & Fumey, D. (2024). Specific Leaf Area and Photosynthesis of Apple Trees Under a Dynamic Agrivoltaic System. *AgriVoltaics Conference Proceedings*, 2. https://doi.org/10.52825/agripv.v2i.999

[3] Sun'Agri. (2023). Agrivoltaics systems: An innovative technique to protect fruit trees from climate change. *Acta Horticulturae*, 1366, 173–186. https://doi.org/10.17660/ActaHortic.2023.1366.20

[4] pv magazine. (2023, August 25). Sun'Agri reveals agrivoltaics performance in heat waves. *pv magazine International.* https://www.pv-magazine.com/2023/08/25/sunagri-reveals-agrivoltaics-performance-in-heat-waves/

[5] pv magazine. (2023, January 2). Mibet commissions 4 MW agrivoltaic project in Japan. *pv magazine International.* https://www.pv-magazine.com/2023/01/02/mibet-commissions-4-mw-agrivoltaic-project-in-japan/

[6] Barron-Gafford, G. A., Pavao-Zuckerman, M. A., Minor, R. L., Sutter, L. F., Barnett-Moreno, I., Blackett, D. T., Thompson, M., Dimond, K., Gerlak, A. K., Nabhan, G. P., & Macknick, J. (2019). Agrivoltaics provide mutual benefits across the food–energy–water nexus in drylands. *Nature Sustainability*, 2(9), 848–855. https://doi.org/10.1038/s41893-019-0364-5

[7] Barron-Gafford, G. *et al.* Benefits of Agrivoltaics Across the Food–Energy–Water Nexus. National Renewable Energy Laboratory (NREL), United States.

[8] Relatório Anual de Progresso – Projeto FruitPV (2024 e primeiro semestre de 2025). *Green & Smart Energy Orchards.* Instituto Nacional de Investigação Agrária e Veterinária (INIAV), Alcobaça, Portugal.

From Matrices to Morphisms: Categorical Programming in Matlab and Octave

by Nelson Martins-Ferreira

Centre for Rapid and Sustainable Product Development, School of Technology and Management
Polytechnic of Leiria
Corresponding email: martins.ferreira@ipleiria.pt

Abstract We develop a categorical and computational framework for programming with finite structures in MATLAB and Octave, aimed at bridging matrix-based computation and morphism-centric reasoning. Morphisms between finite sets are represented concretely using bounded-cardinality models suitable for array-oriented environments, allowing categorical constructions to be realized directly within standard numerical workflows. Building on matrix factorisation mechanisms provided by built-in functions such as unique, we reinterpret common data structures as compositional systems of maps, enabling a uniform treatment of indexing, adjacency, and re-labeling. This perspective supports the expression of geometric and topological relations on purely combinatorial data, while remaining compatible with familiar matrix operations. We introduce categorical programming as a programming paradigm for MATLAB and Octave, formalize its underlying categorical model, and provide categorical interpretations of selected built-in functions. Several illustrative applications are presented, including a categorical approach to the re-indexation of directed graphs.

1 Introduction

From Categorical Programming to Coinductive Semantics: A Lineage of Functional Thought The history of functional programming is closely linked to category theory, which provides a principled framework for reasoning about computation, types, and semantics. A particularly rigorous formulation of this connection appears in Tatsuya Hagino's 1987 thesis, *A Categorical Programming Language* [3], which introduces CPL — a language founded entirely on categorical principles. In CPL, data types and control structures are modeled as instances of F, G-dialgebras, yielding a view of computation grounded in algebraic structure rather than syntactic constructs.

Hagino's work demonstrated that category theory is not merely descriptive but constructive, offering concrete guidance for software design with clear implications for compositionality, modularity, and correctness. What remains less explored is how these ideas translate to numerical, array-based environments that lack native support for typed functional abstraction, yet dominate scientific and engineering computation.

From Hagino to McBride: The Rise of Intensional Semantics The categorical tradition continued with Conor McBride's influential work, notably his 2009 paper *Let's See How Things Unfold* [4], which advances the semantic treatment of infinite data structures. While Hagino emphasized structural abstraction, McBride shifted attention to intensional aspects of computation, focusing on principled reasoning about coinductive types.

By introducing a refined separation between data and codata, McBride exposed limitations of naive coinduction and clarified the distinction between construction and observation. This perspective has had lasting impact on type theory and language design, particularly in systems concerned with unfolding behavior and intensional equality.

Modern Descendants of Intensional Reasoning Subsequent work has translated these semantic insights into practical language mechanisms, largely within dependently typed or effect-aware systems. In *A Type and Scope Safe Universe of Syntaxes with Binding* [5], Allais et al. study syntax with binding through structural recursion, providing generic tools for reasoning about variable-binding constructs in a manner consistent with intensional semantics.

Similarly, the language Frank [6], developed by Lindley, McBride, and McLaughlin, integrates effects and coeffects via a bidirectional type system grounded in categorical semantics. This approach reinforces the practical relevance of intensional reasoning in language design. Finally, *Observational Equality, Now!* [7] advances a notion of equality based on observable behavior rather than syntactic identity, reflecting a broader shift toward semantically meaningful equivalences.

Motivation and Scope of This Work While these developments are rooted in advanced functional languages and type-theoretical frameworks, this work argues that their core insights — compositionality, morphism-centric design, and intensional structure — can be realized concretely in numerical, array-based environments such as Matlab and Octave. Widely used in engineering and applied sciences, these platforms provide an opportunity to explore categorical reasoning outside traditional functional settings.

In particular, a subsequent article will present a categorical, functionally inspired approach to implementing discrete Laplacian operators and the heat method, a standard technique in geometry processing for approximating geodesic distances. This work treats the heat method as both an inspiration and a test of its versatility in addressing concrete, real-world problems. Our contribution is both theoretical and practical: we examine how categorical reasoning guides design decisions and provide concrete MATLAB or Octave code demonstrating a functional programming style in numerically sensitive settings, such as the discretization of partial differential equations on meshes.

From Array Programming to Finite-Set Morphisms At first glance, MATLAB and OCTAVE appear far removed from the categorical and functional traditions discussed above. Their programming model is centered on arrays, indices, and numerical linear algebra rather than on explicit notions of types, morphisms, or compositional semantics. Nevertheless, many core operations in these environments implicitly manipulate structured maps between finite sets, encoded concretely through vectors and matrices.

In particular, common data representations such as index vectors, adjacency lists, and sparse matrices can be understood as concrete realizations of morphisms in the category of finite sets, possibly enriched with multiplicities. Operations that convert between list-based and matrix-based representations—such as the interplay between `sparse` and `find`—exhibit a form of structural duality that mirrors categorical constructions, even though they are typically motivated by efficiency rather than abstraction.

This observation motivates a shift in perspective: instead of treating matrix-based programming as purely numerical, we interpret it as a morphism-centric computation model, where composition, factorization, and reindexation play a central role. By making these structures explicit, we can recover many of the benefits associated with categorical programming—clarity, compositionality, and semantic transparency—within a setting that remains fully compatible with standard Matlab/Octave workflows.

The following section develops this viewpoint concretely, beginning with sparse matrix representations of graphs and showing how familiar programming patterns can be reinterpreted as operations on finite-set morphisms.

2 Advanced Programming with Matlab and Octave

One of the most powerful features in Matlab and Octave for handling large-scale data structures efficiently is the use of sparse matrices. The `sparse` function provides a compact and memory-efficient representation of matrices with predominantly zero entries. This functionality becomes especially valuable in fields such as graph theory, numerical linear algebra, and computational simulations where most interactions are localized or where only a small subset of entries carry information.

A particularly important application of sparse matrices is in the representation of graphs. Consider a graph encoded as a pair of vectors `[d, c]`, where each vector represents a map from a finite set of edges A to vertices in a set B. That is, the vectors `d` and `c` can be viewed as functions $d, c : A \to B$, mapping each edge to its domain (source vertex) and codomain (target vertex), respectively. This encoding allows us to construct the adjacency matrix R of the graph using the sparse function:

```
R = sparse(d, c, 1);
```

This statement creates a sparse matrix R, where the entry $R(i, j)$ represents the number of edges going from vertex i to vertex j. If the graph does not contain parallel edges (i.e., multiple edges between the same pair of vertices), then this sparse representation acts as a binary adjacency matrix, indicating the presence or absence of an edge. However, if the graph includes parallel edges, then the value $R(i, j)$ corresponds to the count of edges between the vertices i and j, effectively transforming the adjacency matrix into a weighted edge-count matrix.

An elegant and useful duality exists between the functions `sparse` and `find`. Specifically, given a sparse matrix R, we can extract its structure using:

```
[d, c] = find(R);
```

This operation retrieves the row and column indices of the non-zero entries of R, effectively recovering the original graph structure encoded in the vectors `d` and `c`. In this sense, `sparse` and `find` serve as inverse operations:

```
[d, c] = find(R);        % Extracts edge list
R = sparse(d, c, 1);     % Reconstructs adjacency
                         matrix
```

The equivalence is exact in the absence of parallel edges. In cases where multiple edges exist between the same pair of vertices, the matrix constructed via `sparse(d, c, 1)` will record the edge multiplicity instead of just connectivity, thereby enriching the representation.

This methodology provides a natural and efficient framework for computing key properties of graphs. For instance, the number of outgoing edges from each vertex can be computed by summing along rows:

```
out_degree = sum(R, 2);
```

Similarly, the number of incoming edges per vertex is obtained by summing along columns:

```
in_degree = sum(R, 1);
```

These operations, built atop sparse matrix handling, scale well with large graphs and form the backbone of graph algorithms in Matlab and Octave.

Interestingly, it turns out that in some situations, building the full adjacency matrix R may not be needed. For example, instead of computing:

```
out_degree = sum(R, 2);
in_degree = sum(R, 1);
```

a more efficient alternative, which avoids constructing R altogether, is to use the following expressions:

```
out = sparse(d, 1, 1);
in  = sparse(c, 1, 1);
```

These lines count how many times each vertex appears as a source (in `d`) and as a target (in `c`), thus giving the out-degree and in-degree vectors, respectively. This approach is often computationally faster and uses less memory, especially for very large graphs.

In conclusion, the `sparse` function, in tandem with `find`, offers a robust mechanism to switch between list-based and matrix-based representations of graphs. This enables concise expression of graph-theoretic operations while maintaining computational efficiency and clarity.

3 A Mixed Category of Indexing Sets and Complex Vector Spaces

We define a category $\mathcal{M}$ whose objects and morphisms are constructed to model interactions between indexing sets (as used in MATLAB-style notation) and complex vector spaces.

Objects The objects of the category $\mathcal{M}$ are of two kinds:

- **Natural number objects:** Each natural number $n \in \mathbb{N}$ is regarded as an object, representing the finite indexing set $\{1, 2, \ldots, n\}$, commonly denoted in MATLAB as `1:n`. Moreover, 0 will represent the empty set.
- **Vector space objects:** For each $n \in \mathbb{N}$, the complex vector space $\mathbb{C}^n$ is also considered an object in the category, representing n-dimensional row vectors over the complex numbers.

Morphisms There are two kinds of morphisms, determined by the source and target types:

- **Indexing morphisms:** A morphism from a natural number object n to another natural number object m is a pair (m, v), where v is a vector of length n with entries in $\{1, \ldots, m\}$, i.e., $v \in \{1, \ldots, m\}^n$. The morphism is interpreted as a mapping from the indexing set `1:n` to `1:m`, assigning to each $i \in \{1, \ldots, n\}$ the value $v(i) \in \{1, \ldots, m\}$.
- **Linear morphisms:** A morphism from $\mathbb{C}^n$ to $\mathbb{C}^m$ is given by a complex $n \times m$ matrix $M \in \mathbb{C}^{n\times m}$, interpreted as a linear transformation $M : \mathbb{C}^n \to \mathbb{C}^m$.

Composition and Identity

- **Indexing morphisms:** Given morphisms $(m, v) : n \to m$ and $(k, w) : m \to k$, their composition is defined as $(k, w(v))$, where $w(v)$ is the vector obtained by applying the indexing vector w to the entries of v elementwise, following MATLAB-style indexing: for each $i = 1, \ldots, n$,

  $$w(v)(i) = w(v(i)).$$

  This corresponds to the composition of discrete functions represented as index vectors.

- **Linear morphisms:** Given matrices $M \in \mathbb{C}^{n\times m}$ and $N \in \mathbb{C}^{m\times k}$, interpreted with domain and codomain defined as:

  $$\mathrm{dom}(M) = \mathrm{size}(M, 1), \quad \mathrm{cod}(M) = \mathrm{size}(M, 2),$$

  the composition $P = N \circ M$ is defined when the codomain of M matches the domain of N, i.e., $\mathrm{size}(M, 2) = \mathrm{size}(N, 1)$. The result is a matrix $P \in \mathbb{C}^{n\times k}$ computed with matrix multiplication as:

  $$P = M * N.$$

  This ordering is consistent with MATLAB's left-to-right evaluation of matrix products and the adopted convention for domain and codomain assignment.

- **Identity morphisms:**
  - For a natural number object n, the identity morphism is (n, id_n), where $\mathrm{id}_n = [1, 2, \ldots, n]$.
  - For a vector space object $\mathbb{C}^n$, the identity morphism is the identity matrix $I_n \in \mathbb{C}^{n\times n}$.

Remarks This category $\mathcal{M}$ mixes discrete and linear structures, allowing interactions between MATLAB-style indexing and linear algebraic operations. Such a framework could be useful for formalizing data transformations, reshaping, and matrix computations commonly found in scientific computing environments.

Mixed Morphisms: Indexing Sets to Vector Spaces In addition to indexing morphisms and linear morphisms, we introduce a third type of morphism representing a map from a natural number object (indexing set) to a complex vector space. This kind of morphism is useful to model row-wise data or collections of vectors.

- **Mixed morphisms:** A matrix $F \in \mathbb{C}^{n\times m}$ is interpreted as a morphism from the indexing object n (i.e., the set `1:n`) to the vector space $\mathbb{C}^m$:

  $$F : n \to \mathbb{C}^m.$$

  In this interpretation, the i-th row of F corresponds to the image of the index i, meaning that $F(i,:) \in \mathbb{C}^m$ is the vector assigned to the i-th element of the domain. Thus, F is a function from the indexing set `1:n` to vectors in $\mathbb{C}^m$, or more formally:

  $$i \mapsto F(i,:) \in \mathbb{C}^m.$$

- **Composition:** Given a vector-valued function represented by $F \in \mathbb{C}^{n\times m}$ (a morphism $n \to \mathbb{C}^m$) and a linear morphism $A \in \mathbb{C}^{m\times k}$ (a morphism $\mathbb{C}^m \to \mathbb{C}^k$), their composition is given by matrix multiplication:

  $$F * A \in \mathbb{C}^{n\times k},$$

  which represents the composed morphism:

  $$n \to \mathbb{C}^k.$$

  This is consistent with MATLAB's row-wise matrix multiplication semantics.

- **Composition with indexing morphisms:** Given a matrix $F \in \mathbb{C}^{n\times m}$ representing a morphism from the indexing object n to the vector space $\mathbb{C}^m$, and an indexing morphism $(n, v) : k \to n$, their composition is defined by row selection:

  $$F(v,:) \in \mathbb{C}^{k\times m},$$

  which yields a new matrix representing a morphism from k to $\mathbb{C}^m$. In MATLAB terms, this corresponds to applying the index vector v to the rows of F. That is, for each $i = 1, \ldots, k$, the i-th row of the result is:

  $$(F(v,:))(i,:) = F(v(i),:) \in \mathbb{C}^m.$$

  This composition is denoted as:

  $$(n, v) \circ F = F(v,:) : k \to \mathbb{C}^m.$$

This third morphism type provides a bridge between indexing sets and vector spaces, enabling interpretation of structured data (e.g., rows of a matrix) as a map into a vector space.

4 Categorical Products and Coproducts in `Matlab-Octave`

In this section, we develop categorical constructions of *products* and *coproducts* within the category **Matlab–Octave**, recall that objects represent finite sets of size $n \in \mathbb{N}$, and morphisms are modeled as index-valued vectors. We first define coproducts (disjoint unions) and then products (Cartesian products), with both formal exposition and practical MATLAB/Octave implementations. We restrict our analysis to the objects modeling finite sets.

Coproducts: Disjoint Union via Concatenation Let $n_1, n_2 \in \mathbb{N}$ be two objects. Their categorical coproduct is defined as:

$$n_1 \amalg n_2 = n_1 + n_2.$$

The canonical injections are given by:

$$\iota_1 : \{1, \ldots, n_1\} \to \{1, \ldots, n_1 + n_2\},$$
$$\iota_2 : \{1, \ldots, n_2\} \to \{1, \ldots, n_1 + n_2\},$$

with implementations:

```
iota1 = 1:n1;
iota2 = n1 + (1:n2);
```

Given morphisms $f : n_1 \to n_B$ and $g : n_2 \to n_B$, represented by index vectors `f` and `g`, the coproduct morphism $h : n_1 + n_2 \to n_B$ is:

```
h = [f; g];   % or h = [f, g];
```

This satisfies the coproduct's universal property:

$$h \circ \iota_1 = f, \quad h \circ \iota_2 = g,$$

verified with:

```
isequal(h(iota1), f)
isequal(h(iota2), g)
```

Products: Cartesian Product via Index Operations The product of n_1 and n_2 is defined as:

$$n_1 \times n_2 = n_1 \cdot n_2.$$

This corresponds to the grid of all pairs (i, j) with $1 \le i \le n_1$, $1 \le j \le n_2$.

The projection morphisms:

$$\pi_1 : n_1 \times n_2 \to n_1, \quad \pi_2 : n_1 \times n_2 \to n_2,$$

are computed via:

```
[pi1, pi2] = ind2sub([n1, n2],
    1:(n1*n2));
```

Given morphisms $f : Z \to n_1$ and $g : Z \to n_2$, the product morphism $h : Z \to n_1 \times n_2$ is:

```
h = sub2ind([n1, n2], f, g);
```

This satisfies:

$$\pi_1 \circ h = f, \quad \pi_2 \circ h = g,$$

and can be verified using:

```
isequal(pi1(h), f)
isequal(pi2(h), g)
```

Concrete Example Let us instantiate with $n_1 = 2$, $n_2 = 3$, and an auxiliary domain $L = 4$. Define:

```
id = @(x) (1:x)';  % Identity function
n1 = 2; n2 = 3; L = 4;

% Coproduct injections
iota1 = id(n1);
iota2 = n1 + id(n2);

% Product projections
[pi1, pi2] = ind2sub([n1, n2],
    id(n1*n2));

% Define morphisms f: L -> n1, g: L -> n2
f = mod(id(L)-1, n1) + 1;
g = mod(id(L)-1, n2) + 1;

% Coproduct morphism
h_coprod = [f; g];

% Product morphism
h_prod = sub2ind([n1, n2], f, g);
```

Verification:

```
isequal(h_coprod(iota1), f)   % Coproduct
    property
isequal(h_coprod(iota2), g)

isequal(pi1(h_prod), f)       % Product
    property
isequal(pi2(h_prod), g)
```

Conclusion In the category **Matlab–Octave**, both coproducts and products are realized using built-in indexing mechanics:

- Coproducts via vector concatenation with index shifting.
- Products via coordinate pairing and conversion between subscripts and linear indices.

This modeling accurately reflects the categorical definitions and offers a robust foundation for further constructions in computational contexts, such as limits, colimits, and exponentials. The construction of products and coproducts relative to the linear objects $\mathbb{C}^n$ is well understood in abelian categories and will not be discussed here. A detailed analysis involving mixed morphisms is deferred to future work.

5 Factorisation of Row Morphisms via the `unique` built-in functionality

As explained before, in the category **Matlab–Octave**, mixed morphisms from a finite indexing object n into a vector space such as $\mathbb{C}^m$ can be represented concretely by an $n \times m$ matrix F. Each row of F corresponds to the image of a distinct element of the domain. When such a morphism is not injective on rows, it can be canonically factorised using the built-in MATLAB/Octave function `unique`.

Given a matrix $F \in \mathbb{C}^{n\times m}$, the call

$$[\mathtt{U}, \mathtt{r}, \mathtt{q}] = \mathtt{unique}(F, \mathtt{'rows'})$$

produces a decomposition of F into:

- `U`: a $k \times m$ matrix of the unique rows of F, ordered lexicographically,
- `r`: a vector of indices $r \in \{1, \dots, n\}^k$ such that $F(r, :) = U$,
- `q`: a vector $q \in \{1, \dots, k\}^n$ such that $F(i, :) = U(q(i), :)$ for each row i.

This decomposition gives rise to a categorical factorisation:

$$F = U \circ q,$$

where:

- $q : n \to k$ is the morphism that indexes each row of F by its corresponding position in the unique matrix U,
- $U : k \to \mathbb{C}^m$ is the morphism that maps each representative index in $\{1, \dots, k\}$ to a distinct row in $\mathbb{C}^m$.

This structure aligns with the standard (split epi)–mono factorisation in category theory. The indexing vector $r \in \{1, \dots, n\}^k$, which identifies the positions of the unique rows in the original matrix, satisfies:

$$q \circ r = \mathrm{id}_k,$$

which establishes r as a right inverse of q, i.e., a *section*. As a consequence,

$$F \circ r = U,$$

which expresses U as the restriction of F to its representative rows. This also confirms that q is a *split epimorphism*, and U is injective on rows (a monomorphism in the categorical sense).

This factorisation is elegantly visualised by the diagram:

$$\begin{tikzcd}
\mathrm{id}(n) \arrow[rr, "F"] \arrow[dr, shift left, "q"] & & \mathbb{C}^m \\
& \mathrm{id}(k) \arrow[ul, shift left, "r"] \arrow[ur, "U"'] &
\end{tikzcd}$$

with $F = U \circ q$, $q \circ r = \mathrm{id}_k$, $U = F \circ r$.

Example To illustrate, consider the matrix:

```
f = -1 * [1 2; 2 2; 3 1; 1 2; 2 2];
[u, r, q] = unique(f, 'rows');
isequal(f, u(q,:))              % verifies F
    = U(q)
id = @(x) (1:numel(x))';
isequal(q(r), id(r))            % verifies
    q(r) = id_k
isequal(u, f(r,:))              % verifies U
    = F(r)
```

This yields the output:

```
f =

  -1  -2
  -2  -2
  -3  -1
  -1  -2
  -2  -2

u =

  -3  -1
  -2  -2
  -1  -2

r =

   3
   2
   1

q =

   3
   2
   1
   3
   2

ans = 1
ans = 1
ans = 1
```

Here, `f` represents the original morphism, `u` the set of distinct output values (unique rows), `q` the morphism $n \to k$, and `r` the section $k \to n$. All verifications of the factorisation properties return `true`.

Conclusion The command `unique(F, 'rows')` provides a concrete and computationally effective realisation of a categorical (split epi)–mono factorisation of morphisms from finite indexing sets to vector spaces. This construction is not only efficient in practice but also faithful to categorical semantics. It allows us to treat data-driven morphisms as abstract maps, and reason about their image and redundancy using standard tools of category theory interpreted in computational terms.

This factorisation method will serve as a building block for further categorical constructions in **Matlab–Octave**, including inverse images, pullbacks, and coequalisers.

6 Pullbacks and inverse images

While internal categories are frequently studied under the implicit assumptions of the existence of pullbacks and a canonical procedure for their construction, these assumptions do not always hold in practical situations in computer science. In many real-world scenarios, such as programming in Matlab and Octave, the existence of pullbacks and a standard method for constructing them cannot be taken for granted.

For instance, in Matlab and Octave, there is a procedure called `ismember` with the signature `[k,p]=ismember(f,m)`, where m is a vector with unique entries, interpreted as a monomorphism, and f is any vector, interpreted as an arbitrary morphism in categorical terms. The procedure outputs a binary vector k of the same size as f and a vector p of the same size as f, containing positions pointing to m such that $m(p) = f$ whenever p has non-zero entries. In addition, it always holds that $m(p(k)) = f(k)$. In Matlab and Octave, vectors can be contracted by precomposing them with binary or logical vectors of the same size. For example, given a vector $f = [1i\ 3i\ -1i\ 3i\ 5i\ 1i\ -1i\ 1i\ 3i]$, precomposing it with a binary vector $k = [1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1]$ results in the contracted vector $f(k) = [1i\ 3i\ 3i\ 5i\ 1i\ 1i\ 3i]$. This contraction property of vectors plays a significant role in the functionality of the `ismember` procedure.

Theoretical Explanation of `ismember` Procedure Using Adjunction Between Subobjects and Characteristic Functions The functionality of the `ismember` procedure in Matlab and Octave can be explained in theoretical terms by considering the adjunction between subobjects and characteristic functions. Instead of directly specifying a subset of elements in vector f, the procedure utilizes a map from the domain of f into the two-element set $\{0, 1\}$. This approach aligns with the concept of characteristic functions and subobjects in category theory.

In the context of computer programming, vectors are not only contracted but also expanded. For instance, consider $m = (1 : 5) * 1i$, which results in $m = [1i\ 2i\ 3i\ 4i\ 5i]$. If we have $p = [1\ 3\ 0\ 3\ 5\ 1\ 0\ 1\ 3]$, then $m(p)$ is not well-defined. However, $p(k)$ is well-defined, and $m(p(k)) = f(k)$. To gain a better understanding of these concepts, interested readers can explore the following list of instructions using online versions of Matlab or Octave. First create vectors f and m with specified values. Define vector p with the given sequence. Attempt to evaluate $m(p)$ and observe the result. Evaluate $p(k)$ and $m(p(k))$ to verify the relationship with $f(k)$. Experimenting with these instructions in a hands-on environment can provide valuable insights into the theoretical underpinnings of the `ismember` procedure and its practical applications in computer programming.

```
octave:1> f=[1i 3i -1i 3i 5i 1i -1i 1i
    3i]; m=(1:5)*1i; [k,p]=ismember(f,m)

k =

  1  1  0  1  1  1  0  1  1

p =
```

```
   1    3    0    3    5    1    0    1    3
octave:2> p(k), f(k), m(p)
ans =

   1    3    3    5    1    1    3
ans =
   0 + 1i    0 + 3i    0 + 3i    0 + 5i    0
       + 1i    0 + 1i    0 + 3i

error: m(0): subscripts must be either
    integers 1 to (2^63)-1 or logicals

octave:3> m(p(k))

ans =

   0 + 1i    0 + 3i    0 + 3i    0 + 5i    0
       + 1i    0 + 1i    0 + 3i

octave:4> isequal(m(p(k)),f(k))

ans = 1
```

Representing Pullbacks Using the `ismember` Procedure in Matlab and Octave The `ismember` procedure in Matlab and Octave can be employed to reproduce the pullback of m along f as depicted in the commutative diagram:

$$\begin{tikzcd} A \times_{\mathbb{C}} B \arrow[r, "p(k)"] \arrow[d, "\mathrm{find}(k)"'] & B \arrow[d, "m"] \\ A \arrow[r, "f"'] \arrow[ur, dashed, "p"] & \mathbb{C} \end{tikzcd} \tag{6.1}$$

In this context, we set $A = (1:9)$ as the domain of f, considered as a map $f\colon A \to \mathbb{C}$, and $B = (1:5)$ as the domain of m, considered as a map $m\colon B \to \mathbb{C}$, with $\mathbb{C}$ representing the set of complex numbers. Given $u\colon X \to A$ and $v\colon X \to B$ such that $fu = mv$, there exists a unique $w\colon X \to A \times_{\mathbb{C}} B$ satisfying $u = kw$ and $v = p(k)u$. This result can be interpreted as follows: since m is a monomorphism, the morphism v can be viewed as a property of u, specifically, the property that fu factors through m. While it would be desirable to represent $A \times_{\mathbb{C}} B$ as a list of elements in B, it is necessary to express it as a subset of A. Although these two requirements are inconsistent, computer scientists have devised an elegant solution by introducing p. While p cannot be considered a morphism in categorical terms due to mp not being well-defined, it serves as a practical solution for representing the object $A \times_{\mathbb{C}} B$ with the two projections $\mathrm{find}(k)$ and $p(k)$. This innovative approach demonstrates the interplay between theoretical concepts in category theory and practical solutions in computer science, showcasing the importance of adapting and extending theoretical frameworks to meet the demands of real-world applications.

7 Categorical Indexation of Directed Graphs via `digraph`

We now illustrate how the ideas from the previous sections (particularly the (split epi)-mono factorisation via MATLAB's `unique` and the ismember function) — can be applied to index the structure of a directed graph (digraph). This is implemented in the function `digraph(source, target, nE)`, which constructs a categorical representation of edges and vertices from geometric data.

Graph Setup Given:

- A number of edges $nE \in \mathbb{N}$,
- Two matrices `source` and `target`, each of size at least $nE \times d$ (where rows represent points in $\mathbb{R}^d$),

we interpret each row `source`(i,:) as the start of a directed edge $i \in \{1, \ldots, nE\}$, and `target`(i,:) as its endpoint. The edges are indexed in the set $1 : nE$.

Factorisation of the Source Map The function applies the row-wise factorisation:

$$[\mathtt{u}, \mathtt{e\_}, \mathtt{d}] = \mathtt{unique(source,\ 'rows')},$$

which decomposes the source matrix as:

$$\mathtt{source} = \mathtt{u(d,:)},$$

yielding a factorisation of the morphism `source` $: nE \to \mathbb{R}^d$ as:

$$\mathtt{source} = u \circ d,$$

where:

- $u \in \mathbb{R}^{k \times d}$ contains the unique source vertices (as rows),
- $d \in \{1, \ldots, k\}^{nE}$ maps edge indices to source vertex indices.

Categorical Structure of the Digraph The set of unique source vertices u is taken as the set of *objects* (vertices), indexed by $B = \{1, \ldots, \mathtt{size}(u, 1)\}$. The edges are represented by the set $E = \{1, \ldots, nE\}$. The function builds a subgraph where:

- An edge $i \in E$ is retained only if `target`(i,:) matches one of the unique source rows. That is, the morphism `target`(i,:) factors through the same indexing as `source`.
- Using `ismember`, the function checks which target vectors lie in the image of u, yielding a boolean mask k and an index vector c such that:

$$\mathtt{target}(i,:) = u(c(i),:), \quad \text{for } i \text{ where } k(i) = 1.$$

- The selected edges are reindexed as:

$$s = d(k), \quad t = c(k),$$

yielding an edge e from vertex $s(e)$ to vertex $t(e)$.

Adjacency and Edge Metadata

- The sparse matrix `Adj` of size $|B| \times |B|$ is constructed so that:

  $$\texttt{Adj}(s(i), t(i)) = E(i),$$

  linking original edge indices to vertex pairs.

- The matrix `Edges` encodes additional edge metadata:

  $$\texttt{Edges}(d(i), E(i)) = \begin{cases} 1 & \text{if } i \text{ is matched via target} \\ -1 & \text{otherwise} \\ 2 & \text{if } i \in \texttt{e\_} \end{cases}$$

where $i \in$ `e_` means that index i has been selected as a canonical source.

Categorical Summary In categorical terms:

- The object set consists of indexed vertices $u : B \to \mathbb{R}^d$.

- The morphisms (edges) are filtered and reindexed using the unique factorisation structure:

  $$\texttt{source} = u \circ d, \quad \texttt{target} = u \circ c.$$

- The sparse matrix `Adj` defines the edge morphisms in terms of their origin and destination in the indexing set B, annotated with their original index $E \in \mathbb{N}$.

- The property $\texttt{source} \circ r = u$, and $d \circ r = \mathrm{id}_B$, implies a split epi–mono structure.

Thus, the function `digraph` constructs a directed graph in which objects correspond to uniquely defined source vertices, and morphisms are represented by indexed edges that satisfy geometric constraints imposed by the `source` and `target` maps. These constraints ensure that the source map s is surjective.

8 A Maltab and Octave implementation for `digraph.m`

The following is a possible implementation of the function `digraph.m` described above.

```
function
    [s,t,u,Adj,Edges]=digraph(source,...
target,nE)
% [s,t,u,Ajd]=digraph(sourse,target)
%
% given a graph structure
    (source,target,nE) in which source
    and target are matrices with the
    same number of columns and at least
    nE lines interpreted as a directed
    graph with nE edges indexed in 1:nE
    and such that an edge i in 1:nE
    starts at vertex source(i,:) and
    ends at vertex target(i,:) with the
    lines in source and target
    interpreted as vectors in euclidean
    space
% Returns the indexed digraph (s,t,u)
    with numel(s)<=nE the number of
    edges i in i:nE with the property
    that there exists at least one j in
    1:nE such that
    source(j,:)==target(i,:), the matrix
    u contains the unique rows of source
    and Adj is the adjacency matrix of
    the digraph (s,t,nA) with
    nA=numel(s)=numel(t) such that
    Adj(i,j)=k has the meaning that k in
    1:nE is the original index of edge k
    in 1:nE in the sense that
    source(k)=s(i) and target(k)=t(j)
% Edges is the adjacency matrix of
    (source,1:nE) where Edges(i,j) can
    be 2, 1,or -1 accordingly to:
    Edges(i,j)=1, the original edge j
    with domain i has the property
    described before ; -1 does not have
    the property above; 2 is a chosen
    edge for vertex index b in
    B=1:size(u,1)
% Example see also the end of file
% source=[1 2; 2 3; 3 4 ]
% target=[3 4; 3 4; 1 2; 3 2]
% [s,t,u,Adj,Edges]=digraph24(source,...
target,3)

 if ~isequal(size(source,2),...
size(target,2)), error('LinksToolbox
    error: source and target must have
    the same number of columns'), end
try
E=(1:nE)';
dom=source(E,:);
cod=target(E,:); disp('yes')
catch
E=(1:min(size(source,1),size(target,1)))';
dom=source(E,:);
cod=target(E,:);
fprintf('Linkstoolbox: the number of
    edges considered was: %d \n ',
    numel(E))
end
```

```
[u,e_,d]=unique(dom,'rows');
[k,c]=ismember(cod,u,'rows');
s=d(k);
t=c(k);
Adj=sparse(s,t,E(k));
cval=zeros(size(k)); cval(k)=1;
    cval(~k)=-1; cval(e_)=2;
Edges=sparse(d,E,cval);

end  % end of function
disp(example)source=[1 2; 2 3; 3 4; 2 3
    ; 3 4]
target=[3 4; 3 4; 1 2; 3 2 ; 2 3]
[s t u
    Adj,Edges]=digraph24(source,target)

% generating a cube
u=fillzeros([0 0 0]',0:1)';
dc=fillzeros([0 0]',1:8)'; d=dc(:,1);
    c=dc(:,2);
uu=u(c,:)-u(d,:);
k=dot(uu',uu')==1;
s=d(k); t=c(k);
source=u(s,:); target=u(t,:);
[s_ t_ u_
    Adj,Edges]=digraph24(source,target);
disp([s t s_ t_])
disp([u, u_])
full(Adj)
```

Referências

[1] N. Martins-Ferreira, *Constructing Platonic solids via stereographic projection with Matlab and Octave implementation* , Scripta-Ingenia **13** (December) (2024) 3–22.

[2] N. Martins-Ferreira, *Internal Categorical Structures and Their Applications* , Mathematics **11** (3) (2023) 660.

[3] Tatsuya Hagino. *A Categorical Programming Language.* PhD thesis, University of Edinburgh, 1987.

[4] Conor McBride. Let's see how things unfold: Reconciling the infinite with the intensional. In *Mathematics of Program Construction*, Springer, 2009.

[5] Guillaume Allais, Robert Atkey, James Chapman, Conor McBride, James McKinna. A Type and Scope Safe Universe of Syntaxes with Binding: Their Semantics and Proofs. *Proc. ACM Program. Lang.*, 2(ICFP):1–30, 2018.

[6] Sam Lindley, Conor McBride, Craig McLaughlin. Do be do be do. *Proceedings of the ACM on Programming Languages*, 2017.

[7] Conor McBride, Thorsten Altenkirch, Wouter Swierstra. Observational Equality, Now! In *PLPV '07: Proceedings of the 2007 ACM SIGPLAN Workshop on Programming Languages Meets Program Verification*, 2007.

Acknowledgment This work received full or partial financial support from the Research and Innovation Agenda for the Sustainability of Agriculture, Food and Agro-industry (Notice No. 09/C05-i03/2021), under project PRR-C05-i03-I-000251 (Fruit-PV), supported by the Recovery and Resilience Plan (PRR) and European Funds NextGeneration EU.

This work also received full or partial financial support the Portuguese Foundation for Science and Technology (Fundação para a Ciência e Tecnologia) FCT/MCTES (PIDDAC) through the following Projects: Associate Laboratory ARISELA/P/0112/2020; UIDP/04044/2020; UIDB/04044/2020; PAMI - ROTEI- RO/0328/2013 (N° 022158); MATIS (CENTRO-01-0145-FEDER-000014 - 3362); Generative.Thermodynamic; by CDRSP and ESTG from the Polytechnic of Leiria.

Topological surfaces and anchored complex links

by Nelson Martins-Ferreira

Centre for Rapid and Sustainable Product Development
School of Technology and Management
Polytechnic of Leiria

Author email: martins.ferreira@ipleiria.pt ;

Abstract This short paper presents a constructive procedure for modeling discrete Riemannian surfaces using the structure of anchored complex links. By encoding surface topology and planar embedding data in a purely combinatorial framework, the approach provides a foundation for defining discrete geometric operators on mesh-free surfaces.

Introduction In this paper, we outline a constructive approach for modeling discrete Riemannian surfaces using the structure of anchored complex links. By combining topological considerations with algebraic representations in the complex plane, complex links provide a flexible framework for encoding both connectivity and geometric information. The proposed procedure highlights how anchoring such links enables a consistent description of surface topology while supporting discrete notions of metric and curvature. This perspective offers a bridge between classical surface theory and discrete geometric models, with potential applications in geometry processing and related areas.

Surfaces as Anchored Complex Links In preparation for the study of discrete Laplacians and heat flow on combinatorial surfaces, we first adopt a suitable categorical and computational representation of a surface. Rather than relying on smooth manifolds or triangulated meshes in the classical sense, we define a surface via a minimal yet expressive discrete structure. Our approach is informed by both topological intuition and computational tractability within the **Matlab–Octave** environment.

Combinatorial Encoding of a Surface We define a *surface* as a combinatorial structure based on a set of directed edges equipped with additional data encoding both local and global topological features. This leads us to the notion of an *anchored complex link* [1, 2].

Definition 1. *An* ***anchored complex link*** *is a tuple* (n_A, g, φ, s) *where:*

- $n_A \in \mathbb{N}$ *is a natural number indexing a finite set of* half-edges*, denoted* $A = \{1, \ldots, n_A\}$*,*
- $g : A \to \mathbb{C}$ *assigns to each half-edge a complex number representing its* direction *(typically of unit norm, encoding angle or orientation),*
- $\varphi : A \to A$ *is an endomap whose orbits encode the cyclic ordering of half-edges around each* face*,*
- $s : A \to B$ *is a surjection onto another finite set* $B = \{1, \ldots, n_B\}$*, indexing the* vertices *of the surface, subject to further coherence conditions detailed below.*

This structure can be visualised diagrammatically as:

$$\begin{tikzcd}
A \arrow[loop left, "\varphi"] \arrow[r, "g"] \arrow[d, "s"'] & \mathbb{C} \\
B &
\end{tikzcd}$$

Interpretation of the Structure The components of the anchored complex link admit the following interpretations:

- Each element $a \in A$ is a *half-edge*, understood as a directed incidence between a vertex and a face.
- The map φ encodes the *face rotation system*: the orbit of a given $a \in A$ under φ gives the ordered sequence of half-edges bounding a single face.
- The map s identifies half-edges that share a common *origin vertex*; in categorical terms, s classifies the orbits of a function $\theta : A \to A$ satisfying

  $$s \circ \theta = s, \quad \theta \circ \varphi \circ \theta \circ \varphi = \mathrm{id}_A.$$

  The function θ is such that the composition $\theta \circ \varphi$ acts as a *dart reversal*, pairing each half-edge with its inverse (i.e., the same edge with reversed orientation).
- The function $g : A \to \mathbb{C}$ provides embedding data, assigning to each half-edge a complex number interpreted geometrically as a vector or angle in the plane. This data can be used to construct a geometric realisation of each face.

Taken together, these elements define a directed multigraph endowed with additional structure on both faces and vertices. In this sense, the tuple (n_A, g, φ, s) generalises and refines the notion of a combinatorial surface—similar in spirit to ribbon graphs or rotation systems—while remaining amenable to an index-based computational implementation.

Categorical and Computational Remarks Recall from the previous article that sets of the form $\{1, \ldots, n\}$ are uniformly encoded in our framework by the natural number n, treated as an object in the category **Matlab–Octave**. Morphisms between such objects are index vectors or, more generally, functions implemented as arrays.

Thus, in the context of anchored complex links:

- The set of half-edges A is represented by the object $n_A \in \mathbb{N}$,
- The maps φ, θ, s are represented by index vectors $\varphi, \theta \in A^A$ and $s \in B^A$,
- The embedding data g is a complex-valued vector $g \in \mathbb{C}^{n_A}$.

This approach is fully compatible with the categorical programming model developed in the previous article: the use of natural numbers as indexing objects, together with morphisms realised as explicit vectors, permits a seamless implementation of topological structures as data objects within MATLAB or Octave.

Applications and Further Connections The concept of an anchored complex link plays a central role in computational geometry. For instance, as shown in [1], this structure suffices to encode the symmetries and connectivity of classical polyhedral surfaces, including Platonic solids and their generalisations.

Furthermore, this representation is particularly well suited for defining discrete differential operators such as the Laplacian. In the articles that follow, it will exploit the structure of an anchored complex link to define adjacency relations and weights for the combinatorial Laplacian, the heat method, and more general diffusive processes, following a strategy of discretising geometric flows on a mesh-free topological substrate.

A Note on Indexing Conventions Although objects in our category are formally represented as natural numbers n, it is often convenient in this geometric setting to identify each object with the set $\{1, \ldots, n\}$ itself. This more concrete viewpoint facilitates the description of orbits, adjacency relations, and local connectivity, and simplifies the exposition of algorithms operating on indexed collections of edges and vertices.

We will therefore move freely between these equivalent views—treating objects either as abstract cardinalities or as explicit index sets—while maintaining categorical consistency in the representation and composition of morphisms.

A Functor into Topological Manifolds Given an anchored complex link (A, g, φ, s) as detailed above, we now describe a canonical construction of an associated topological surface.

Let Q denote the coequaliser of the pair of morphisms $(1_A, \varphi)$, so that Q indexes the orbits of the endomap φ, and let

$$q \colon A \to Q$$

be the corresponding projection map. Each element $k \in Q$ thus corresponds to a face of the surface.

For each $k \in Q$, we define a planar polygon $F(k) \subset \mathbb{C}$ as the convex hull of the points

$$\{g(x) \in \mathbb{C} \mid x \in A,\ q(x) = k\},$$

with vertices ordered according to the cyclic order induced by φ. We then consider the disjoint union

$$X = \bigsqcup_{k \in Q} F(k).$$

The topological surface is obtained by gluing the faces $F(k)$ along their edges: vertices are identified whenever $s(x) = s(y)$, and the corresponding edges are identified according to the dart-reversal structure induced by $\theta \circ \varphi$. The resulting quotient space carries a natural topology, yielding a (possibly singular) two-dimensional manifold associated functorially to the anchored complex link.

Concrete example Consider a simple discretization of the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, consisting of its north pole $p_N = (0, 0, 1)$ together with four points on the equator, $p_1 = (1, 0, 0)$, $p_2 = (-1, 0, 0)$, $p_3 = (0, 1, 0)$, and $p_4 = (0, -1, 0)$.

Let $B = \{p_1, p_2, p_3, p_4, p_N\}$ denote the set of vertices, and consider the four spherical triangles having p_N as a common vertex, together with the square spanned by (p_1, p_2, p_3, p_4) along the equator, as a coarse discretization of the sphere.

This combinatorial surface can be encoded by an anchored complex link with half-edge set

$$A = \{1, \ldots, 16\},$$

where the indices 1 through 12 correspond to the half-edges of the four upper triangular faces, and the indices 13 through 16 correspond to the half-edges of the equatorial square. The remaining structure maps (φ, s, g) are then defined in the obvious way from this incidence data, yielding an explicit anchored complex link representation of the discretized sphere.

Numerical Implementation of the Heat Method

by R. Almeida[1], J. Duque[1], M. Gaspar[1,2], N. Martins-Ferreira[2], R. A. P. Perdigão[3]

(1) Department of Mathematics, Faculty of Science and Technology, University of Beira Interior
(2) Centre for Rapid and Sustainable Product Development, School of Technology and Management, Polytechnic of Leiria
(3) Meteoceanics Institute for Complex System Science, IUC Physics of Complex Coevolutionary Systems & Fluid Dynamical Systems, Washington, DC, USA
Corresponding email: martins.ferreira@ipleiria.pt

Abstract We implement the heat method on surfaces defined by triangulated meshes. The approach provides a flexible and algebraically robust foundation for discrete differential operators, facilitating applications in geometry processing, numerical analysis, and spectral methods. A series of examples, including Platonic solids and quadrilateral meshes, illustrates the expressiveness and computational efficiency of the method.

1 Introduction

In this short note, we present a practical implementation of the heat method for computing distances on surfaces represented by triangulated meshes. The heat method [3] has gained attention for its simplicity, robustness, and efficiency, making it a valuable tool in geometry processing and numerical analysis. By relying on discrete differential operators, our implementation offers a flexible algebraic framework that adapts naturally to a wide range of surface geometries. Through illustrative examples, including Platonic solids and quadrilateral meshes [1, 2], we highlight both the expressiveness of the method and its strong computational performance, demonstrating its suitability for applications ranging from spectral analysis to mesh-based simulations.

2 Preliminary Steps

We now present the preliminary steps toward a practical implementation of the discrete Laplacian and the associated gradient-divergence framework for heat flow simulation, culminating in the so-called *heat method* [3]. The MATLAB code below illustrates this process on a surface derived from a Platonic solid [1], using the combinatorial formalism introduced in [2], and discussed further in the articles *Topological Surfaces* and *From Matrices to Morphisms* in this issue.

2.1 Setup and Anchored Complex Link

The following code uses the dodecahedron (Platonic solid with 12 faces) as a sample triangulated surface:

```
[V,T] = platonic_solids(4); % Dodecahedron
[g,phi,s] = tri2link(T,V); % Construct the anchored complex link
```

We extract the face structure from ϕ (`phi`), and reorder the data accordingly:

```
[face, ford] = orbits(phi);
[~, sortedbyfaces] = sortrows([face ford]);
g = g(sortedbyfaces);
phi = phi(sortedbyfaces);
s = s(sortedbyfaces);
```

The edge directions and vertex positions are reconstructed from the complex directions:

```
dir = zeros(size(phi));
pos = zeros(size(phi));
dir(phi) = exp(cumsum(g));
pos(phi) = cumsum(dir);
```

This also means that from `pos` we recover `dir` and `ace` as `ace = dir(phi)./dir` and `dir = pos(phi)-pos`.

2.2 Polygonal Geometry

To compute intrinsic geometric quantities like face area and perimeter, we define:

$$\texttt{vectorarea}(z,\phi) = \frac{1}{2}\sum_{a\in A} \operatorname{Im}(\overline{z(a)}z(\phi(a)))).$$

This is implemented in MATLAB as:

```
vectorarea = @(z,phi) 1/2 *
    sparse(orbits(phi),1, ...
      real(z).*imag(z(phi)) -
        real(z(phi)).*imag(z));
Area = vectorarea(pos, phi);
Peri = sparse(face, 1, abs(dir));
```

A rescaling factor $v \in \mathbb{R}^F$ is chosen to normalize the geometry:

$$v = \frac{\sqrt{P^2 + 32\pi A} - P}{4A},$$

where $P = \text{Peri}$ and $A = \text{Area}$. This ensures that the scaled polygon approximates a unit disk in both area and perimeter.

2.3 Mass and Gradient Operators

From the complex edge directions $g = \log(\texttt{ace})$, where `ace = dir(phi)./dir` and `dir = pos(phi)-pos`, we extract:

$$r = e^{\mathrm{Re}(g)}, \quad t = \mathrm{Im}(g),$$

and define the edge-wise mass:

$$\texttt{mass}(a) = \frac{\pi - t(a)}{6}\,(1 + r(a) + r(a)^2).$$

```
r = exp(real(g));
t = imag(g);
mass = (pi - t)/6 .* (1 + r + r.^2);
M = sparse(s(phi), 1, mass);
```

A family of gradient approximations is implemented, with ‘grad1‘, ‘grad2‘, and ‘grad21‘ representing successive factorizations:

```
grad1 = @(u) sparse(face, 1, (u(s(phi))
    - u(s)) .* dir);
grad2 = @(G) G(face) ./ dir;
grad21 = @(u) grad2(grad1(u)); %
    equivalent to composed gradient
```

Alternative formulations (‘grad3‘, ‘grad4‘) depend only on the logarithmic edge encoding g, which may be useful for compressed storage or symbolic computations:

```
grad=@(u)
    (u(s(phi(phi)))-u(s(phi))).*exp(g) +
    (u(s(phi))-u(s));
grad3=@(u)
    (u(s(phi(phi)))-u(s(phi))).*dir(phi)
   + (u(s(phi))-u(s)).*dir;
grad4=@(u)
    (u(s(phi(phi)))-u(s(phi))).*exp(g) +
    (u(s(phi))-u(s));
```

Further research is needed to examine these and other potential formulations.

2.4 Divergence and Laplacian

Several divergence definitions are proposed. A physically meaningful one uses the flux across edge boundaries:

```
flux = @(X) sparse(s(phi),1,X .* mass ./
    exp(1i*(t+pi)/2)) ./ M;
% diverg = @(X) real(flux(X));
```

A cotangent-weight Laplacian can be computed via:

$$W_{uv} = \sum_{a\in A,\, s(a)=u,\, s(\phi(a))=v} r(a)\cdot\cot(t(a)/2),$$

and symmetrized to ensure self-adjointness:

```
W = sparse(s, s(phi), r .* cot(t/2));
W = (W + W') / 2;
L = W - diag(sum(W)); % Laplacian matrix
```

Note that this formula differs slightly from the classical cotangent operator [3].

2.5 Conclusion and Next Steps

The proposed implementations have the potential to realize the heat method on general triangulated (or polygonal) surfaces within the formalism of anchored complex links. However, the key operators—gradient, divergence, and Laplacian—require further analysis to determine which formulations are most appropriate in concrete applications. Nevertheless, our approach, developed in alignment with a categorical and combinatorial perspective, aims to satisfy the following key properties:

- Mass is defined consistently from local angular geometry.
- The gradient is defined on faces and lifted to edges.
- The divergence collects fluxes back to vertices, ensuring conservation.
- The Laplacian is symmetric and encodes both combinatorics and geometry.

This provides a foundation for applying the heat method to compute:

- Geodesic distance approximations,
- Spectral analysis of the Laplacian,
- Smoothing and diffusion processes on combinatorial surfaces.

Future work may investigate further factorizations of the Laplacian operator or its spectral decomposition, continuing our categorical and computational development of discrete differential geometry.

3 Variational and Finite Element Formulation of the Heat Equation

To make the previously discussed combinatorial and categorical constructions concrete, we now formulate the heat equation in a variational (weak) sense. This provides a bridge from the abstract operators—gradient, divergence, and Laplacian—to practical computations on planar and polygonal complexes, paving the way for a finite element discretization compatible with the heat method [3].

3.1 Weak Formulation in Real Coordinates

Let $\Sigma \subset \mathbb{R}^2$ be a bounded polygonal domain equipped with the standard Euclidean metric and area element $dx\,dy$. We consider the classical heat equation

$$\partial_t u - \Delta u = f \quad \text{in } \Sigma \times (0, T],$$

where

$$\Delta u = \partial_x^2 u + \partial_y^2 u.$$

For simplicity, we impose homogeneous Dirichlet boundary conditions

$$u|_{\partial\Sigma} = 0.$$

Weak Formulation

Let

$$V := H_0^1(\Sigma)$$

denote the Sobolev space of real-valued functions with square-integrable first derivatives and vanishing trace on the boundary.

Multiplying the equation by a test function $\varphi \in V$, integrating over Σ, and applying integration by parts (using the boundary condition) yields

$$\int_\Sigma \partial_t u\, \varphi\, dx\, dy + \int_\Sigma \nabla u \cdot \nabla \varphi\, dx\, dy = \int_\Sigma f\, \varphi\, dx\, dy.$$

Introducing the bilinear form

$$a(u, \varphi) := \int_\Sigma \nabla u \cdot \nabla \varphi\, dx\, dy,$$

the weak problem reads

$$(\partial_t u, \varphi)_{L^2(\Sigma)} + a(u, \varphi) = (f, \varphi)_{L^2(\Sigma)} \qquad \forall \varphi \in V.$$

3.2 Piecewise Planar Polygonal Complex

We now assume that Σ is obtained from a finite collection of planar polygonal faces embedded in $\mathbb{R}^2$, with pairs of edges identified so that the resulting space is connected and each vertex admits a well-defined star neighborhood.

Each face carries the standard Euclidean metric inherited from the plane. The resulting space is therefore piecewise Euclidean: the metric is flat in the interior of each face and may have cone-type singularities at vertices.

Let $\mathcal{F}$ denote the set of faces and $\mathcal{V}$ the set of vertices.

Functions $u : \Sigma \to \mathbb{R}$ belong to $H^1(\Sigma)$ if their restriction to each face lies in $H^1(F)$, they are continuous across identified edges, and the sum of the facewise Dirichlet energies is finite.

The weak formulation remains

$$(\partial_t u, \varphi)_{L^2(\Sigma)} + \int_\Sigma \nabla u \cdot \nabla \varphi\, dA = (f, \varphi)_{L^2(\Sigma)},$$

where all integrals are understood as sums over faces.

Finite Element Approximation

Let $\mathcal{T}_h$ be a conforming triangulation of Σ with vertex set V_h. We choose a finite-dimensional subspace

$$V_h \subset V$$

spanned by basis functions

$$\{\psi_i\}_{i \in V_h},$$

indexed by the vertices of the triangulation. We assume that

- each ψ_i is continuous on Σ,
- each ψ_i is supported on the star of vertex i,
- each ψ_i is affine on every triangle $T \in \mathcal{T}_h$.

We approximate the solution by

$$u_h(x, y, t) = \sum_{i \in V_h} U_i(t)\, \psi_i(x, y).$$

Testing against ψ_j gives the semi-discrete system

$$M \frac{dU}{dt} + AU = F,$$

where

$$M_{ij} = \int_\Sigma \psi_i \psi_j\, dx\, dy, \qquad A_{ij} = \int_\Sigma \nabla \psi_i \cdot \nabla \psi_j\, dx\, dy.$$

The mass matrix M is symmetric positive definite, and the stiffness matrix A is symmetric positive semidefinite.

3.3 Complex Reformulation

We now reinterpret the same construction using complex coordinates. Identify $\mathbb{R}^2$ with $\mathbb{C}$ via

$$z = x + iy, \qquad dA = dx\, dy = \frac{i}{2}\, dz \wedge d\bar{z}.$$

Define the complex derivatives

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

Then

$$\Delta = 4\, \partial_z \partial_{\bar{z}},$$

and for sufficiently smooth u,

$$|\nabla u|^2 = 4\, |\partial_{\bar{z}} u|^2 = 4\, |\partial_z u|^2.$$

Hence the Dirichlet energy admits the equivalent representation

$$\int_\Sigma |\nabla u|^2\, dx\, dy = 4 \int_\Sigma |\partial_{\bar{z}} u|^2\, dA.$$

The heat equation becomes

$$\partial_t u - 4\,\partial_z \partial_{\bar{z}} u = f,$$

and the weak formulation can be written equivalently as

$$\int_\Sigma \partial_t u\,\overline{\varphi}\,dA + 4\int_\Sigma \partial_{\bar{z}} u\,\overline{\partial_{\bar{z}}\varphi}\,dA = \int_\Sigma f\,\overline{\varphi}\,dA.$$

The corresponding finite element matrices are therefore

$$M_{ij} = \int_\Sigma \psi_i \overline{\psi_j}\,dA, \qquad A_{ij} = 4\int_\Sigma \partial_{\bar{z}}\psi_i \overline{\partial_{\bar{z}}\psi_j}\,dA,$$

which represent exactly the same operators as in the real formulation, expressed in complex coordinates.

Connection with the Heat Method

To approximate geodesic distance from a source vertex i_0, the heat method proceeds as follows:

1. Solve the short-time heat equation
$$(M + tA)U = M\delta_{i_0}.$$
2. Compute the piecewise-constant facewise gradient ∇u_h, and normalize it to obtain a unit vector field $X = -\frac{\nabla u_h}{|\nabla u_h|}$.
3. Compute the discrete divergence of X at vertices and solve the Poisson problem
$$A\phi = \operatorname{div}(X).$$
4. The solution ϕ approximates the intrinsic geodesic distance from i_0.

Because the geometry is piecewise Euclidean, all differential operators are defined locally on faces, while global consistency is enforced by continuity across identified edges.

4 Conclusion

We have presented initial steps toward a categorical and computational framework for discrete differential geometry, centered on the representation of surfaces via anchored complex links. In particular, surfaces are modeled as collections of polygonal faces—possibly arising from triangulations—arbitrarily placed in the plane, with edges identified consistently to define a coherent combinatorial structure.

This framework enables the definition of discrete geometric operators, including gradient, divergence, and Laplacian, directly on combinatorial data structures. These operators respect the intrinsic geometry of the surface through carefully constructed mass and directional fields derived from local angular information.

Finally, the implementation of the heat method within this categorical formalism demonstrates the power of the approach to unify topological, geometric, and numerical insights. It provides a solid foundation for further developments, including geodesic computation, spectral analysis, and discrete analogues of classical differential geometric structures.

Referências

[1] N. Martins-Ferreira, *Constructing Platonic solids via stereographic projection with Matlab and Octave implementation* , Scripta-Ingenia **13** (December) (2024) 3–22.

[2] N. Martins-Ferreira, *Internal Categorical Structures and Their Applications* , Mathematics **11** (3) (2023) 660.

[3] K. Crane, C. Weischedel, M. Wardetzky (2017). The heat method for distance computation. Communications of the ACM, 60(11), 90-99.

[4] T. Needham, *Visual Complex Analysis*, Clarendon Press, Oxford, 1997.

Acknowledgements - N.M.F. acknowledges the Portuguese Foundation for Science and Technology (Fundação para a Ciência e Tecnologia) FCT/MCTES (PIDDAC) through the following Projects: Associate Laboratory ARISELA/P/0112/2020; UIDP/04044/2020; UIDB/04044/2020; PAMI - ROTEI- RO/0328/2013 (N° 022158); MATIS (CENTRO-01-0145-FEDER-000014 - 3362); Generative.Thermodynamic; by CDRSP and ESTG from the Polytechnic of Leiria. This work also received full or partial financial support from the Research and Innovation Agenda for the Sustainability of Agriculture, Food and Agro-industry (Notice No. 09/C05-i03/2021), under project PRR-C05-i03-I-000251 (FruitPV), supported by the Recovery and Resilience Plan (PRR) and European Funds NextGeneration EU.

- R.A.P.P. acknowledges the MICSS Flagships MR-220617 and MR-200628 (QITES), and also the European Union supporting the Horizon Europe grant 101074004 (C2IMPRESS).

- M.G. acknowledges support from the Portuguese Foundation for Science and Technology (Fundação para a Ciência e Tecnologia) under grant 2022.14868.BD.

- R.A. and J.D. acknowledge funding from the Portuguese Foundation for Science and Technology (Fundação para a Ciência e Tecnologia) under UID/00212/2025.

A minimal energy dimagma structure for complex plane discretization

by N. Martins-Ferreira(1) and R. A. P. Perdigão(2)

(1) Centre for Rapid and Sustainable Product Development
School of Technology and Management
Polytechnic of Leiria
(2) Meteoceanics Institute for Complex System Science, IUC Physics of Complex Coevolutionary Systems & Fluid Dynamical Systems, Washington, DC, USA
Author email: martins.ferreira@ipleiria.pt ; perdigao@meteoceanics.org

Abstract This paper continues the investigation initiated in *A short introduction to dimagmas and their applications to physics*, where dimagmas were introduced as elementary algebraic structures capable of capturing essential features of both mathematical and physical systems. Building on that conceptual framework, we now focus on a more concrete and constructive problem, namely the discretization of the complex plane through a dimagma endowed with a minimal energy principle. Motivated by the alternative notion of dimagma energy introduced previously, we seek algebraic configurations that minimize this energy while preserving the ability to generate complex-valued structures. The resulting minimal energy dimagma provides a natural and efficient discretization of the complex plane, grounded in simple binary operations rather than additional axioms. This approach offers a unifying perspective that links algebraic simplicity, geometric representation, and physical interpretation, and it further clarifies how dimagma-based constructions can encode both structural regularity and energetic optimality in discrete models.

1 Introduction

Modeling complex systems, especially entailing scaling, self-organization, symmetry breaks, emergence of self-similar structures (e.g. coherent dissipative structures such as in convection, anomalous diffusion), often require rather formally involved and laborious methodological frameworks, such as in the scope of fractional-order dynamical systems, nonlinear statistical mechanics, just to name a few.

One of the key issues in this regard pertains the scale propagation of local perturbations such as anomalous micro-physical diffusion, often elusive to direct detailed observation, all the way up to easily discerned macro-physical features that can then be more aptly characterized in thermodynamic or kinematic-geometric form as macro-scale patterns. In physical terms, this is often represented as exponential growth of such local perturbations, corresponding to the entropy production mechanism known as deterministic chaos.

At the same time, it is important to reconcile local discrete phenomenology such as in quantum mechanical context, with global mesoscale and macro-scale features typically observed as discrete patterns in the midst of a global continuum space.

In formal mathematical terms, this requires a construct aptly bridging discrete with continuous structures, without a priori assumptions of structural dynamic invariance and symmetry.

This is particularly relevant in such fields as quantum mechanics, where commutativity is lost (commutator linked to the delocalization underlying what is now known as the uncertainty principle), relativity, where associativity is lost upon sequential rotation in space-time, and relativistic quantum mechanics e.g. in neutrino dynamics, where sequential charge-momentum-flavour operations can lack both commutativity and associativity leading to symmetry breaks (or violations) underlying the prevalence of key neutrino configurations relative to their rotated counterparts.

Traditionally, a continuum would live within the notion of a topology. Recall that a topology on a set X is a family of subsets of X, $\tau \subseteq \mathcal{P}(X)$, whose elements are called *open*, and is such that a finite intersection of open sets is an open set, any arbitrary union of open sets is an open set and both $\emptyset$ and X are open sets. A pair (X, τ) in which X is a set and $\tau \subseteq \mathcal{P}(X)$ is a topology on X is called a topological space.

However, our intent is to treat systems that are not necessarily entailing a continuum but rather with a finite number of discrete elements, while still benefiting from established topological notions traditionally associated with the continua. Therefore, we propose retaining fundamental topological notions whilst removing the axiom of intersection of open sets being an open set. This is necessary since a topological space of a discrete set would degenerate onto a preordered set, lacking some of the richness of a full-fledged topological construct.

For that purpose, we shall introduce a formal bijective mechanism assigning a finite discrete set of indices to a set of objects, in such a way that even if the latter (objects) live in a continuum and are morphed under continnum transformations, the former (indices) can nevertheless entail discrete algebraic operations without loss of information or generality.

As a motivational example, consider k continuous wavefunctions Ψ_k interfering in a continuum spacetime $\mathcal{S}$, through triadic wave resonance. When operating such in the object space (the continuum), the operation is expressed as a convolution integral among said wavefunctions, which is computationally laborious and error prone to perform in a brute-force discretisation. However, by representing said Ψ_k via discrete spectral indices $s(k)$ which can be as simple as frequencies or any quantum number (e.g. in spherical wave theory underlying quantum chemical description of atoms), the wave interference can be computed in the index space (spectral computation), which allows for lossless algebraic operation that then yields the resulting wave without loss of generality and without the need for any approximations. Then the resulting wave Ψ_r emerges from the triadic resonance via the inversion of the object-to-index transformation. This example illustrates how via discrete indices and operations complex interactions among objects living in a continuum can be performed without loss of generality by benefitting from the structure that is hereby generalised.

In the present study we shall focus on a simpler broader example, that of the discretization of the Complex Plane.

The present contribution thus aims to provide a simple formal mechanism to enact such continuum-discrete mapping through discretization of the complex plane, along with its computational implementation.

The default type of variables in Matlab and Octave is rectangular matrices whose entries are complex numbers seen as a pair of floats. This means that each complex number is encoded as a string of 128 bits. It consists of two blocks of 64 bits, each one of them is a floating point that is encoded in the IEEE 754 Double Precision standard.

In this note we are going to propose an alternative representation of complex numbers. First we fix two natural numbers, N and M, then consider the set of indexes from 1 to M^N and associate a complex number to each one of the indexes. By working with the original representation $z(a) \in \mathbb{C}$ for each $a \in (1 : M^N)$ as well as a derived formula $w(a)$ expressed with the complex exponential map we take full advantage of representing a large spectrum of complex numbers within a relative range of indexes, comparable to the standard 128 bits. In addition, for every dimagma structure that can be defined on the set of indexes $1 : M^N$ there is an associated functional which can be interpreted as its energy, by measuring the cumulative differences of performing the operations on the dimagma structure and then evaluating their complex realization or else to evaluate first each repesentation and then to perform the usual operation of addition and multiplication of complex numbers. The dimagma showcasing the least energy possible is the best candidate to the encoding of our representation as an alternative number system to the standard IEEE.

In spite of the disadvantage that the IEEE standard is already implemented in all computers and that changing such standards is always an herculean task, we believe, that the future of computations in physics will take the path here proposed.

2 Discretization Procedure

The discretization procedure shall build from Martins-Ferreira and Perdigão (2024), which can be outlined as follows.

Let M and N be two (fixed) given natural numbers. The number of points in the discretization will be M^N. Define:

$$\begin{aligned} B &= \{1, \ldots, M\} \\ X &= \{1, \ldots N\} \\ A = B^X &= \{a \mid a \colon X \to B\} \end{aligned}$$

Each $a \in A$, being a map from X to B, can be identified with a vector of size N with entries ranging in $1 : M$.

Let us also consider two auxiliary maps f and g. The map $g \colon B \to \mathbb{C}$ is given by

$$g(b) = \begin{cases} 0 & , b = M \\ e^{2\pi i \left(\frac{M-b}{M-1}\right)} & , b < M \end{cases}$$

whereas $f \colon A \to \{0, -1, 1, -2, 2, -3, 3, \ldots\}$ is given by

$$f(x) = \begin{cases} -\frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x-1}{2} & \text{if } x \text{ is odd.} \end{cases}$$

The following step is in accordance to the desire of having a discretization motivated by quantum mechanics. Namely, to have a value R beyond which physical measurements are not meaningful. On the other hand, we also expect the appearance of a small values h which itself is not zero but its square is zero. Now, many attempts have been made in the past to overcome these dificulties and many have fail. We believe that the failure is due to the fact that we have been insisting that the basic operations of addition and multiplication should be associative, commutative, distributuve and etc, but we now see that in the discrete case there is no resonable reason to adhere to such restrictions. In our interpretation, R can be comparable to the radious of the universe while h is comparable to the Planck constant.

We propose the following definition, which are dependent on N and M, or in other words, they depend on the resolution of the discretization. The functions f and g are the auxiliary ones defined before:

$$R = \sum_{x \in X} 3^{f(x)}$$

The reader may be wondering why the number 3, why not 2 or any other? Because 3 has the best properties for combining positive and negative powers of the form 3^n. Moreover, it seems relevant that the following (non-associative but commutative) structure can be defined on the set $\{0, 1, -1\}$, where 0 is a neutral element and a commutative operation defined as follows:

$$\begin{aligned} 1 + (-1) &= 0 \\ 1 + 1 &= 1 \\ (-1) + (-1) &= -1 \end{aligned}$$

This operation is not associative since $1 + (1 + (-1)) = 1 + 0 = 1$, but $(1 + 1) + (-1) = 1 + (-1) = 0$. Thus, the system $(\{0, 1, -1\}, +, 0)$ is not a monoid but it nevertheless possesses interesting algebraic properties. In particular it suggests a reazonable way to define a closed operation on a finite set other than modular arithmetic.

Continuing our definition we put:

$$h = \frac{\sqrt{2}}{2M^N}$$

Once again, the square root is used in order to guarantee that our discretization, as detailed below, does not have repeated complex values for different indexes.

First, we define $z \colon A \to \mathbb{C}$ as

$$z(a) = \sum_{x \in X} g(a(x)) 3^{f(x)}$$

for every $a \in A = B^X$, with f and g as before.

In addition, we define $w \colon A \to \mathbb{C}$ as

$$w(a) = \exp\left(-\Im(z(a)) + i\pi\tau\left(\frac{\Re(z(a))}{R}\right)\right)$$

where $\tau \colon [-1, 1] \to [-1, 1]$ is any smooth monotone map such that

t	$\tau(t)$
-1	$-1 + 2h$
0	$\frac{h}{1+h}$
1	$1 - h^2$

which aligns with the usual convention that the main branch of the exponencial map is defined for angles strictly bigger that $-\pi$ and smaller or equal than π. As we see from the table above $\tau(1) = 1 - h^2$ and assuming that h^2 is zero we recover $\tau(1) = 1$. On the other hand, $\tau(-1) = -1 + 2h$ which is strictly bigger than -1. In addition, the fact that $\tau(0)$ is different from 0 provides for $w(a_0)$ to be different from 1, where $a_0 \in A$ is such that $z(a_0) = 0$. Otherwise we would have $w(a_0) = z(a_1)$ with $a_i \in A$ such that $z(a_1) = 1$.

Finally the complete discretization uses two copies of each index in A, one copy indexes $z(a)$ another copy indexes $w(a)$. Formally, we have a map $F \colon A + A \to \mathbb{C}$ from the coproduct of A with itself to the set of complex numbers, defined as

$$F(u) = \begin{cases} z(a), & \text{if } u = \iota_1(a) \\ w(a), & \text{if } u = \iota_2(a) \end{cases} \tag{2.1}$$

where ι_1 and ι_2 are the two canonical inclusion into the coproduct $A + A$.

As a standard, we propose that for each one of the bit representations: 4bit, 6bit, 8bit, 16bit, 32bit, 64bit, 128bit, 256bit, to find the optimal value of N so that $2M^N$ can be represented with $M \in \{3, 5, 13, 25\}$.

As an illustration we give some examples. For simplicity we are using τ as the identity map.

```
clear, clf, cla
k=0; N=2:4; M=3:5;
for k_N=1:numel(N)
for k_M=1:numel(M)
k=k+1
nX=N(k_N); nB=M(k_M);
% e.g. nX=3, nB=3
X=1:nX;
B=1:nB;
A=fillzeros(zeros(nX,1),B);
nA=size(A,2);
isequal(nA,nB^nX);
f=zeros(1,nX); % inicialize f with zeros
f(2:2:end)=-(2:2:nX)/2; % f(x) if x is
    even
f(1:2:end)=((1:2:nX)-1)/2; % f(x) if x
    is odd
g=[exp(2*pi*1i*(nB-B(1:end-1))/(nB-1)),
    0];

R=sum(3.^f);
h=sqrt(2)/(2*nA);
z=@(a)
    sum(g(a).*3.^repmat(f’,1,size(a,2)));
w=@(a)
    exp(-imag(z(a))+1i*pi*(real(z(a))/R));
F=[z(A), w(A)];

figure(1)
subplot(numel(N),numel(M),k)
plot(F,’.’),
title(fprintf(’N=%d, M=%d’,[nX nB]))
end
end
```

Finally, we can look at the best dimagma structure on the discrete set of indeces $A + A$ which minimizes the energy in the following sense.

Given any dimagma structure on A, say $(A, \circ, \cdot)$, define its *energy*, denoted by $E(A + A, \circ, \cdot) \in \mathbb{C}$, via the formula

$$E(A + A, \circ, \cdot) = \alpha + i\beta,$$

where:

$$\alpha = \sum_{u, v \in A + A} (F(u) + F(v) - F(u \circ v)))$$

whereas

$$\beta = \sum_{\substack{u,v \in A+A \\ F(u\cdot v)\neq 0}} \left(\frac{F(u)F(v)}{F(u\cdot v)} - 1\right)$$

and moreover $F\colon A+A \to \mathbb{C}$ is defined as in equation 2.1.

Note that this approach to represent the complex plane has advantages and disadvantages. It requires an addition and multiplication table instead of an algorithm to compute the operations using decimal, or binary, expansions.

3 Dimagma Structure Transport

A natural way to transport the dimagma structure from the complex plane to the discrete set of indices is to discretize the complex plane using a pair of maps, z and z^{-1}, and to define a structure called a *Dimagma* over the set $A = \{1, 2, \ldots, nA\}$. The set A represents discrete indices that correspond to points in the complex plane, and the binary operations of addition $(+)$ and multiplication $(\cdot)$ are defined using the discretized maps. These operations give rise to the *Plus* and *Times* tables, which are essential components of the Dimagma structure.

Relative to the function z defined in the previous section, we note that this will be slightly modified in its construction in the sense that instead of having $3^{f(x)}$ we take $log(3)f(x)$, in order to then be able to apply the exponential to obtain, at the same time, the real and imaginary parts, as can be seen below.

```
% The bijection between A and B^X
s2i=@(nB,s) (s-1)*(nB.^((1:size(s,2))'-1))+1; % nX=size(s,2)
i2s=@(i,nB,nX) mod(floor((i(:)-1)./nB.^((1:nX)-1)),nB)+1;
% using the fact that i./s expands i to the size of s

nB=4, nX=2, nA=(nB+1)^nX
s=i2s(1:nA,nB+1,nX);
mask=logical(mod(s,nB+1));
x=log(3)*repmat((1:nX)-floor(nX/2)-1,size(s,1),1);
y=2i*pi*s/nB;
g=sum(exp(x+y).*mask,2);
plot(g,'.')
%pause
% the map z:A-->Complex for a given i in A

z=@(i) reshape(sum(exp(log(3)*((1:nX)-floor(nX/2)-1)+...
2i*pi*i2s(i,nB+1,nX)/nB).*logical(mod(i2s(i,nB+1,nX),nB+1)),2),size(i));

% check that z(i) is defined as desired
isequal(g,z((1:nA)'))

% for the retraction of z, zinv:Complex-->A
zinv=@(w) reshape(argminmax(z((1:nA)')-w(:).','min'),size(w));

plot(z(1:nA),'.')
pause

[x,y]=ndgrid(linspace(-3^(nX/2),3^(nX/2),nA));
imagesc(zinv(x+1i*y))
max(abs(g)), max(x(:))
pause

% Having z and zinv we compute the Plus and Times tables as
[i,j]=ndgrid(1:nA);
Plus=zinv(z(i)+z(j));
Times=zinv(z(i).*z(j))
imagesc(Plus)
pause
imagesc(Times)
```

```
pause


% Energy
E_=abs(abs(z(i)+z(j)−z(Plus)+1i∗abs(z(i).∗z(j)./z(Times)−1)));
E=sum(E_(isfinite(E_(:))))
```

3.1 Discretization Using z and z^{-1}

The key idea behind the discretization is to create mappings that transform a continuous complex plane into a finite set of points. This is achieved by defining two functions:

- $z : A \to \mathbb{C}$ is the map that takes an element from the discrete set A and maps it to a complex value in the plane.
- $z^{-1} : \mathbb{C} \to A$ is a pseudo-inverse map, which takes a complex value and returns the corresponding discrete index from A.

The map $z(i)$ is defined as:

$$z(i) = \sum \exp\left(\log(3)\cdot\left((1:nX)-\text{floor}\left(\frac{nX}{2}\right)-1\right)\right. \\ \left. +2i\pi\cdot\frac{i2s(i,nB+1,nX)}{nB}\right)\cdot L$$

where $L = \text{logical}\left(\text{mod}\left(i2s(i,nB+1,nX),nB+1\right)\right)$.

This function discretizes the complex plane using exponential functions involving both real and imaginary components, where nB is the base of the discretization and nX is the number of dimensions. The exponential terms determine how the points in the complex plane are spaced and scaled.

The pseudo-inverse map $z^{-1}(w)$ is defined as:

$$z^{-1}(w) = \text{reshape}\left(\text{argminmax}\left(z((1:nA)')+ ... \right.\right. \\ \left.\left. -w(:),'\,min'\right), \text{size}(w)\right)$$

Here, z^{-1} finds the index in the set A that corresponds to a given complex value w, thus reversing the discretization process.

Instead of resorting to brute force to calculate the index that lies closest to a given point w, the following procedure can be considered. Since for each index i, z(i) is obtained through a sum with different weights, namely those in the form $3^{f(x)}$, we subtract, relative to w, the modulus value of $3^{f(x)}$ and evaluate what argument from the indices with that absolute value best approximates to the given value for w. This means that instead of having a search space in the order of N^M, we have an order of $N * M$.

3.2 Plus and Times Operations

Now we define the binary operations of addition $(+)$ and multiplication $(\cdot)$ on the set A. As previously explained, these operations are inherited from the discretization of the complex plane, as follows:

The Plus table is defined as the operation of addition on the discrete set A:

$$\text{Plus(i,j)} = z^{-1}(z(i)+z(j))$$

This operation adds two elements of the set A by first mapping them to their corresponding complex values via z, then performing complex addition, and finally mapping the result back to the discrete set using z^{-1}.

The Times table is defined as the operation of multiplication on the discrete set A:

$$\text{Times(i,j)} = z^{-1}(z(i)\cdot z(j))$$

This operation multiplies two elements of the set A by first mapping them to complex values via z, performing complex multiplication, and mapping the result back to the discrete set using z^{-1}.

Both operations are visualized using the `imagesc` function, which shows how the set A behaves under addition and multiplication, providing insights into the structure of the discrete complex plane.

3.3 Energy Calculation in the Discrete Structure

The next step is to compute the energy of the system. The energy function E is computed for every pair of indexes $E_{i,j}$ as:

$$\left(z(i)+z(j)-z(\text{Plus(i,j)})+1i\cdot\left(\frac{z(i)\cdot z(j)}{z(\text{Times(i,j)})}-1\right)\right)$$

This function calculates the absolute values of the differences and ratios of the mapped values, accounting for the differences between addition and multiplication operations.

The final energy value is computed by summing the valid (finite and real) values of $E_{i,j}$:

$$E = \sum_{i,j} E_{i,j}\ (\text{only finite values of } E_{i,j})$$

This energy calculation is crucial for analyzing the behavior and structure of the discretized complex plane.

3.4 Visualization and Interpretation

Finally, the project provides visualizations of the mappings and operations:

- The points g in the complex plane are plotted as a scatter plot, showing their distribution.
- The behavior of z^{-1} on a grid of complex numbers is visualized using `imagesc`, allowing for the examination of the inverse mapping in the complex plane.
- The Plus and Times operations are visualized through their corresponding tables, showing how the set A behaves under these binary operations.

These visualizations and computations collectively demonstrate the discrete structure formed by the set A, the mappings z and z^{-1}, and the binary operations Plus and Times, all of which arise from the discretization of the complex plane.

3.5 Conclusion

This approach to discretizing the complex plane via the maps z and z^{-1} and defining a Dimagma structure with binary operations allows for a systematic exploration of the complex plane's properties in a discrete setting. By combining algebraic structures like addition and multiplication with geometric representations of the complex plane, this project provides insights into both the algebraic and geometric aspects of the discretized complex system.

Acknowledgements - N.M.F. acknowledges the Portuguese Foundation for Science and Technology (Fundação para a Ciência e Tecnologia) FCT/MCTES (PIDDAC) through the following Projects: Associate Laboratory ARISELA/P/0112/2020; UIDP/04044/2020; UIDB/04044/2020; PAMI - ROTEI- RO/0328/2013 (N° 022158); MATIS (CENTRO-01-0145-FEDER-000014 - 3362); FruitPV; Generative.Thermodynamic; by CDRSP and ESTG from the Polytechnic of Leiria. This work also received full or partial financial support from the Research and Innovation Agenda for the Sustainability of Agriculture, Food and Agro-industry (Notice No. 09/C05-i03/2021), under project PRR-C05-i03-I-000251 (FruitPV), supported by the Recovery and Resilience Plan (PRR) and European Funds NextGeneration EU.

- R.A.P.P. acknowledges the MICSS Flagships MR-220617 and MR-200628 (QITES), and also the European Union supporting the Horizon Europe grant 101074004 (C2IMPRESS).

From Representation to Realization: Structural Dynamics Across Science, Art, and Technology

by Ana Isabel Marques(2,3) and Nelson Martins-Ferreira(1,3)

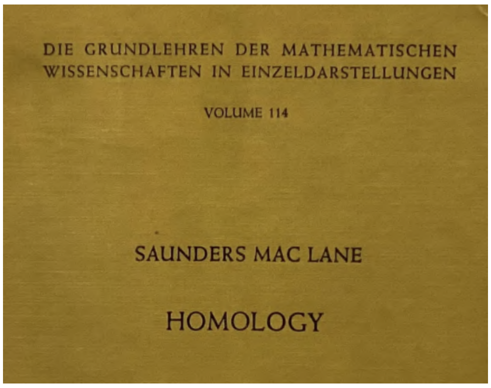
(1) Centre for Rapid and Sustainable Product Development
(2) Margarida Losa Institute for Comparative Literature
(3) School of Technology and Management
Polytechnic of Leiria
Author email: martins.ferreira@ipleiria.pt ; amarques@ipleiria.pt

Abstract This paper examines the shared structural dynamics that underlie scientific inquiry, artistic practice, and technological production, focusing on the interplay between representation, creativity, and material realization. Drawing on perspectives from philosophy of science, aesthetics, and contemporary research in physics, chemistry, and the arts, it argues that knowledge and creation unfold through interdependent processes of imagination, structural invention, and reproducible realization. Rather than treating theory and practice, or art and science, as distinct domains, the paper highlights common patterns of organization, such as symmetry, invariance, and transformation, that guide both conceptual understanding and material intervention. By tracing how abstract forms are conceived, stabilized, and embedded within collective practices and infrastructures, the study reveals a persistent tension between individual insight and shared responsibility, possibility and constraint. This integrated perspective not only clarifies how disciplines generate and transmit meaning, but also foregrounds the ethical and cultural implications of how ideas are shaped, enacted, and reproduced in the world. The paper ultimately proposes a framework for understanding human creativity and knowledge as structured, relational, and fundamentally situated within social and material contexts.

1 Introduction

Understanding the interplay between representation, structure, creativity, and material realization is central to both scientific inquiry and artistic practice. Across disciplines, human activity navigates a tension between seeing and transforming, between abstract conception and concrete realization, between individual insight and collective responsibility. Historical and contemporary thinkers have emphasized different facets of this dynamic: from B. J. Caraça's reflections [1] on the dialectic of individual and collective in the formation of culture, to Lemaitre's critique of narcissism and empathy in modern scientific institutions. These perspectives converge on a common insight: knowledge is never merely descriptive, nor is creation ever entirely unconstrained; rather, understanding and action unfold through structured processes that intertwine imagination, invariance, invention, and reproduction.

Recent advances in physics, chemistry, and digital fabrication, alongside developments in painting, sculpture, and architecture, provide striking examples of this interplay. In theoretical sciences, the invention of form—through symmetry principles, relational structures, or morphogenetic reasoning—creates conceptual frameworks that are intelligible across domains. In the arts, the transmutation of materials, gestures, and spatial configurations demonstrates analogous processes of structural innovation. Meanwhile, experimental techniques, instruments, and technological infrastructures stabilize and reproduce these forms, mediating the gap between conceptual insight and material reality. Across these fields, the dynamic duality between invention and realization, imagination and execution, reflects a broader pattern: the persistent tension and co-dependence between what we conceive and what we enact, between individual insight and collective embedding, between theoretical possibility and practical effect.



DIE GRUNDLEHREN DER MATHEMATISCHEN
WISSENSCHAFTEN IN EINZELDARSTELLUNGEN
VOLUME 114

SAUNDERS MAC LANE
HOMOLOGY

By examining these processes through an integrated lens, it becomes possible to trace the pathways through which human understanding operates. Representation and structural analysis reveal the invisible patterns that guide thought; creative transformation generates new possibilities; material practice embeds these possibilities in the world. In highlighting these interconnected layers, we can better appreciate how disciplines traditionally separated by method and medium—mathematics and painting, chemistry and architecture, theory and practice—share common principles of organization, transformation, and reproduction. Such an approach emphasizes not only the cognitive and technical dimensions of knowledge, but also its ethical, social, and cultural ramifications, inviting reflection on the ways in which intellectual, artistic, and technological practices shape both human perception and the shared world.

2 Imagology and Homology via Category Theory

Imagology and homology theory can be brought under a single conceptual framework if both are understood through the language of category theory. Hugo Dyserinck's intervention in the 1960s was to insist that national and cultural "images" and "mirages" in literature are not empirical descriptions of real peoples but textual constructs that operate within and across literary traditions. What imagology studies, therefore, is not cultures as such, but the relations by which cultures are represented, mediated, distorted, and transmitted in discourse. This shift already implies a categorical perspective: the primary objects of study are not isolated entities but structured systems of relations.

One may formalize this by treating imagology as a field of study concerned not with real referents (materiality), but with their representations, which encompass both individual and collective self- and hetero-images (e.g. "France viewed as", "Germany viewed as") and whose morphisms are representations of one such position within another culture's literature. These morphisms include travel writing, narrative tropes, stereotypes, and genre conventions. They are composable: a representation of Germany in French literature may be taken up and transformed again in English literature, yielding a composite morphism. Identity morphisms correspond to auto-images, self-representations that are no less constructed than hetero-images. In this category, morphisms are generally not invertible, and distortion is not an error but a structural feature. This captures Dyserinck's claim that images and mirages are intrinsic literary phenomena rather than deviations from sociological accuracy.

Homology theory, as axiomatized by Eilenberg and Mac Lane, provides a parallel model. In homology, one does not study spaces directly, but passes from spaces to chain complexes and then applies functors to extract invariants. The central question is not what a space looks like point by point, but what structural features persist under continuous deformation. Homology replaces geometric intuition with functorial invariance. This is the decisive move that allows very different spaces to be compared within a single formal framework.

Imagology can be understood in exactly the same way. Instead of topological spaces, one begins with discursive formations. A corpus of texts representing one culture through another can be decomposed into strata: explicit descriptions, narrative motifs, genre patterns, and broader ideological frameworks. These strata can be organized into a chain complex, where the boundary maps represent tensions, contradictions, or rearticulations between levels. Dyserinck's distinction between images and mirages fits naturally here: images correspond to lower-level chains that are relatively anchored in textual convention, while mirages function like higher-level cycles that do not resolve into empirical reference but nevertheless structure meaning.

From this perspective, imagological analysis becomes a homological procedure. One may define imagological homology groups that measure which representations persist across literary production, translations, adaptations, and reception in both the source and target contexts. A national stereotype that reappears across centuries and literary systems corresponds to a nontrivial homology class. A fleeting exotic fantasy that appears in a single text but leaves no trace elsewhere corresponds to a boundary and therefore vanishes under homological analysis. What imagology seeks, like homology, is not truth but persistence under deformation. As a matter of fact every image is, by definition, ontologically distinct from the reality to which it refers and is invariably a construction, thus eluding the notions of "truth" and "falsehood" (cf. Machado/Pageaux, 1988) [5].

Category theory provides the unifying metalanguage that makes this parallel precise. Text production, translation, adaptation and reception are naturally modeled as functors between categories of representations.

Methodological rigor in imagology amounts to respecting functoriality: one must not confuse objects and morphisms, or collapse distinct representational levels into empirical claims about real cultures. Historical shifts in national imagery can be modeled as natural transformations between such functors, explaining how large-scale changes in representation occur coherently across many texts without requiring a change in the underlying cultural "object."

Even classical imagological phenomena acquire categorical interpretations. Stereotypes arise as colimits, in which many heterogeneous representations are collapsed into a single simplified figure. Critical reconstruction, by contrast, aims at limits, seeking the most constrained representation compatible with all available textual evidence. Imagology thus studies the tension between these two categorical tendencies in cultural discourse.

Seen in this light, imagology is best described as the homology theory of intercultural representation. Just as homology abstracts away from metric and local detail to capture structural invariants of spaces, imagology abstracts away from individual texts to capture structural invariants of representation. Dyserinck's contribution lay in intuiting a shift from textual content to extratextual representations, thereby enriching literary studies through their intersection with other fields of knowledge. Although the formal language of category theory in the sense of Mac Lane and Eilenberg had not yet entered literary studies, Dyserinck's role in imagology is structurally analogous to Mac Lane's role in homology: both clarified the nature of the objects their fields are truly concerned with, and demonstrated why relations and invariants matter more than isolated facts.

At a broader scale, this logic of imagological construction can be observed in what Yuval Noah Harari [4] describes as forms of collective narcissism, whereby nations or civilizations come to view themselves as uniquely central, indispensable, or exemplary in the history of humankind. Such narratives do not merely reflect historical self-confidence; they function as powerful extratextual representations that organize memory, justify political projects, and stabilize collective identity by projecting an image of exceptional importance. In this sense, national narcissism operates analogously to individual narcissism: it privileges self-referential narratives, amplifies symbolic visibility, and marginalizes relational or comparative perspectives. This collective level provides an essential intermediate step for understanding how narcissism migrates from large-scale cultural imaginaries into more localized institutional settings, preparing the ground for an analysis of its manifestation within the epistemic culture of modern science.

Collective self-representations that are strongly narcissistic — typical of nations that view their culture as historically central — tend to accumulate symbolic capital within the academic field. Such contexts function as reference centers of legitimacy, which produces a structural advantage for scholars trained within them, whose work is more easily recognized and validated. Conversely, researchers from culturally peripheral fields often encounter barriers to recognition, since their position in the hierarchy of symbolic power weakens the initial reception of their contributions. In contexts marked by a more negative collective self-image, this asymmetry may also foster an overinvestment in internationalization strategies, a strong dependence on external validation, and a relative devaluation of local forms of recognition.

Aware of these validation codes, more strategically positioned researchers may deploy the internationalization of their work — sometimes in redundant or weakly integrated ways — as a mechanism for converting external recognition into reinforced prestige within their domestic academic field.

Lemaitre book and the issue of narcissism Bruno Lemaitre's *An Essay on Science and Narcissism* provides a concrete case study of how a particular personality dimension — narcissism — functions as an image within a specific institutional discourse, namely academia. Lemaitre, an experienced life-scientist, argues that modern academic science is characterized by an increasing prevalence of individuals whose primary orientation is not towards collaborative inquiry or communal values but towards self-advancement, visibility, and dominance. Narcissism, in the social-psychological sense, can be described as a tendency to "get ahead" rather than "get along," an obsession with status and admiration, an inflated self-image and a strong drive for personal distinction that often overshadows collective norms of cooperation and critical integrity. This rise of narcissism, visible in strategic networking, media performance, and career-oriented behaviors rewarded by contemporary research evaluation systems, functions as a *cultural image* of what it means to be a successful scientist in Western academia, even as it often contradicts the idealized image of the scientist as modest and truth-oriented. Such narcissistic behavior is not merely individual pathology; it is a *discursive construct* that shapes expectations, interactions, incentives and hierarchies within the scientific community. The image of the narcissistic scientist thus becomes a kind of stereotype embedded in academic texts, practices, evaluation criteria and socialization processes, one that affects not only how individuals behave but how the institution as a whole is represented to itself and to the outside world. At the same time, Lemaitre's account emphasizes that narcissism does not exist in isolation but in duality with empathy, communal responsibility and meticulous care for the integrity of research. The meticulous scientist, low in narcissism but high in a sense of community, is often marginalized within such a system because the institutional "image" of success privileges self-promotion over empathetic collaboration. This tension between narcissistic self-projection and empathetic scholarly engagement can itself be interpreted imagologically as a structural dichotomy or polarity in the representation of scientific subjects and careers: the self-centered elite persona versus the community-oriented researcher persona.

Based on the terminology of G. Siebenmann [9], who develops the concept of identity in its various dimensions, the self-centered subject tends to prioritize a self-image detached from the collective otherness, whereas the community-oriented researcher structures a self-image as an integral part of the collective otherness.

Within the categorical framework developed earlier, these competing orientations can be seen as distinct objects in a category of *academic representations*, with morphisms corresponding to the ways in which narratives about science, success and identity are propagated, translated, and adjusted across texts, evaluations, and career pathways. The persistence of the narcissistic image across institutional contexts — in hiring practices, funding competitions, publication cultures, and popular media portrayals of scientists — resembles a nontrivial homological class: a representational feature that survives deformation under the "maps" of academic procedures and discourse transformations and that continues to structure the social field. Empathy, by contrast, often fails to form such persistent classes because it is less visible, less rewarded, and therefore more likely to be treated as a boundary in the chain complex of academic representations; it is a feature that collapses in the face of systemic incentives that favor self-promotion. The concrete cases Lemaitre discusses — manipulative dominance, reward structures that amplify narcissistic traits, and the sidelining of meticulous community-oriented practice — illustrate how the *image of narcissism* is not only a descriptive notion but a functional agent in the ongoing self-construction of academic culture. From an imagological perspective, the interplay between narcissism and empathy, in Lemaitre's analysis, reveals the ways in which institutional images shape both internal practice and external understanding of science. This duality between narcissism and empathy, when seen categorically, highlights how certain representational invariants, i.e., isotopic lines (narcissistic success images) persist and propagate through academic culture while others (empathetic, collaborative science) fail to achieve the same structural robustness, suggesting that a deeper examination of these images — and the incentives and narratives that support them — is necessary to understand not only individual behaviors but the evolving image of science itself [2].

Bento de Jesus Caraça's 1933 lecture "A Cultura Integral do Indivíduo" In his 1933 lecture *A Cultura Integral do Indivíduo*, the Portuguese mathematician and intellectual Bento de Jesus Caraça addresses the crisis of his time by reflecting on the relationship between the individual and the collective within social life. Delivered amid the tumultuous rise of fascism, economic depression and political instability in Europe, B. J. Caraça's intervention laments the escalating conflict between what he calls the *individual principle* and the *collective principle*, a dialectic that he sees as defining human history. According to B. J. Caraça, the individual and collective forces are in perpetual tension: the individual may contribute creative energy to social development but, left unchecked, can devolve into egoistic domination that undermines the collective good. This polarity, described in terms of the struggle between ego and community, resonates closely with the imagological duality between narcissism and empathy that emerges in Bruno Lemaitre's critique of contemporary academia.

From an imagological and identity-based perspective, this relates to whether the individual priorizes individuality or collective belonging in the construction of the self-image. B. J. Caraça's conception of the cultural individual anticipates this duality by showing how the *image of the individual* within political and social discourse functions both as a driver of progress and as a potential source of social fragmentation. The image of human greatness, of heroic autonomy detached from collective responsibility, becomes a recurring trope or isotopic line — not only as an intellectual abstraction but as a socially circulated representation that shapes expectations about personal achievement, leadership and virtue under the conditions of crisis. In imagological terms, the *individual* appears as a representational object with its own morphisms into social texts, political rhetoric, educational practices and institutional norms, just as the *collective* appears as a counter-image that stands for solidarity, cooperation and shared purpose. These two images are not merely competing ideas but are embedded in the very structure of cultural discourse, influencing how people shape their identities, how institutions marshal narratives of success and purpose, and how social roles are imagined across time. In B. J. Caraça's analysis, the dominance of the individual over the collective leads to repeated cycles of social upheaval, because the image of the autonomous self, while enabling creative breakthroughs, also facilitates the elevation of egoistic interests that subvert communal aims. This mirrors the problem Lemaitre identifies in academia, where a culture valorizing narcissistic self-promotion over empathetic collaboration reproduces institutional structures that reward visibility and status at the expense of collective scholarly values. Through the lens of imagology, the tension between the individual and the collective becomes a structured polarity of images circulating in texts and practices: the image of the self-assertive individual and the image of the empathetic collaborator, each with its own set of morphisms and transformations across genres, institutions and historical contexts. Categorical thinking allows us to model these as persistent representational patterns — the "individual-as-ego" image or euphoric self-image forming a nontrivial homological class that continues to propagate through political and academic discourses, while the "collective-as-empathy" image often remains a boundary that fails to form stable invariants or isotopic lines under the dominant maps of institutional incentives. B. J. Caraça's lecture, when read alongside Lemaitre's critique, thus illuminates how deeply the imagological polarity of narcissism and empathy is rooted in broader cultural narratives about human identity, social purpose, and the role of institutions. This contributes a historical depth to the contemporary analysis of academic culture, suggesting that the institutional image of the individual has long been a central structuring element of modern social imaginaries, with concrete effects on how communities structure and perceive the self, the other, and collective identity [1].

In conclusion B. J. Caraça's historical opposition between the individual principle and the collective principle can be read, within the same categorical framework, as an early formulation of the narcissism–empathy polarity operating at the level of imagological homology. The image of the autonomous individual, celebrated as creative and exceptional, corresponds to a representational class that persists across political, cultural, and institutional deformations, forming a stable homological invariant that reappears in modern academia as the narcissistic figure of success. The counter-image of the collective, grounded in solidarity and shared responsibility, aligns with empathy as a representational orientation that repeatedly fails to stabilize under dominant institutional mappings, remaining fragile and often reduced to a boundary rather than a persistent class. In this sense, both B. J. Caraça's analysis and Lemaitre's critique describe the same structural asymmetry: the categorical conditions under which images of individual self-assertion or euphoric self image propagate robustly, while images of empathetic collectivity struggle to achieve comparable invariance within modern cultural and academic imaginaries.

From epistemic posture to operative form A similar imagological–homological mechanism can be observed if one turns not to a historical figure or an institutional discourse, but to the contemporary figure of the artificial intelligence system itself. In this case, the "subject" under consideration is neither an individual psyche nor a social class, but a technical artefact whose mode of operation nonetheless produces stable images, expectations, and representational invariants within culture. As an author interacting with such a system, I am confronted not merely with a tool, but with an image of cognition: an entity perceived as neutral, objective, efficient, and structurally rational, often contrasted implicitly with human subjectivity, bias, and affect. This image of AI functions imagologically in much the same way as the narcissistic or collective images analysed earlier: it circulates through discourse, media, institutional practices, and everyday interaction, shaping how knowledge, authority, and creativity are represented.

From a homological perspective, what is striking is the persistence of certain structural features or isotopic lines across radically different representations of AI. Whether described in popular media, technical documentation, ethical debates, or direct user interaction, the AI is repeatedly mapped as a system that operates through relations rather than intentions, through pattern recognition rather than meaning, through transformation of existing material rather than originary creation. These features form a nontrivial homological class: they survive deformation across contexts, languages, and narratives. At the same time, other potential images — for instance, AI as collaborative partner, dialogical counterpart, or sociotechnical hybrid embedded in human practices — tend to remain unstable, often collapsing into boundaries rather than persisting as robust representational invariants.

Examining the internal operation of such systems reinforces this structural reading. What appears externally as "intelligence" is, internally, a process of mapping inputs onto high-dimensional relational spaces, identifying regularities, weighting associations, and reproducing patterns that have proven statistically stable across vast corpora. Meaning, intention, and empathy do not appear here as intrinsic properties, but as effects projected onto the system through interaction. The AI does not "understand" in a human sense; it preserves, transforms, and recombines structures. In categorical terms, it operates almost exclusively at the level of morphisms: translating between representations, preserving certain invariants while discarding others, and stabilizing patterns that recur under repeated mappings.

This makes the AI a particularly revealing object for the transition from imagology to homology. The images projected onto it — objectivity, neutrality, efficiency, or even authority — are imagological constructs, while the underlying operation of the system exemplifies homological reasoning in a pure form: the extraction and preservation of relational structure independent of semantic depth or experiential grounding. The gap between these two levels — between the cultural image of intelligence and the structural reality of pattern transformation — mirrors, in technical form, the earlier tension between narcissism and empathy, individual and collective. It is precisely by understanding this gap that one can move beyond mere representation and structural analysis toward the question of transformation itself: how new forms arise, how structures are actively reconfigured, and how invention becomes possible. It is at this point that the discussion must turn from homology to ingenia.

3 Praxis and Ingenia

Ingenia Ingenia, seen as the art of transmuting forms and processes, extends the structural and representational work of imagology and homology into the realm of conceptual creation. It concerns the invention of coherent relational architectures that are not mere reflections of existing reality, but generative frameworks capable of producing new forms, processes, and possibilities. Henri Poincaré, in *Science and Hypothesis*, exemplifies this approach in theoretical physics and mathematics: scientific theory is a creative act, a disciplined construction of relations and structures that renders phenomena intelligible without simply cataloging empirical facts. Ingenia, in this sense, is the capacity to organize, connect, and transform concepts according to internal coherence and structural insight. Hermann Weyl's analysis of symmetry further situates ingenia across domains: symmetry acts as a trans-domain operator linking physics, mathematics, and the arts, providing invariants under transformation that guide the generation of new forms while preserving structural intelligibility. In chemistry, Linus Pauling's *The Nature of the Chemical Bond* demonstrates ingenia concretely, as abstract concepts such as orbitals and bonding structures transmute experimental data into unified, predictive architectures, revealing the hidden relational patterns of matter. René Thom's work on structural stability and morphogenesis provides a formal language for ingenia that applies simultaneously to physics, biology, and art: transformation is treated not as arbitrary change, but as the evolution of forms constrained by internal stability and relational logic, producing configurations that are intelligible across domains. Within the categorical framework developed earlier, ingenia can thus be interpreted as the creation of new morphisms and transformation rules that extend existing structures while preserving coherence. It bridges homological invariance and material praxis, generating conceptual patterns that may later be stabilized and reproduced, whether in experimental physics, chemistry, or artistic creation, and forming the conceptual ground upon which praxis can enact the reproduction of reality.

Praxis Praxis, the art of reproducing reality, complements ingenia by translating conceptual structures into material, operational, and performative form. While ingenia invents and transfigures forms and processes, praxis enacts them in the world, stabilizing structures through interaction with physical, social, and technological constraints. Peter Galison, in *Image and Logic*, demonstrates how theory, instrumentation, and practice coevolve: experimental setups, measurement devices, and procedural protocols are not merely passive conduits for testing theory, but active mediators that stabilize reality and enable reproducible knowledge. Norbert Wiener's *Cybernetics* extends this understanding, emphasizing feedback, control, and the recursive reproduction of behavior, showing that praxis is not linear execution but dynamic, self-correcting realization of structural patterns. Contemporary technological practices illustrate this vividly: Mario Carpo, in *The Second Digital Turn*, examines how 3D printing, parametric design, and digital fabrication reproduce conceptual forms in concrete materiality, linking theoretical invention with precise operational execution. Classical practice, exemplified in Vitruvius' *De Architectura*, unites theory, technical mastery, and material realization, demonstrating that architecture embodies the continuum from ingenia to praxis: a conceptual design only becomes reality through skilled manipulation of materials and techniques. From a socio-historical perspective, Karl Marx in *Capital* highlights how praxis is constrained by existing material structures, production processes, and labor relations: the reproduction of reality is never neutral, but occurs within systemic frameworks that shape the possibilities of action. Within the categorical framework established earlier, praxis can be understood as the realization of morphisms and structures generated by ingenia: it operationalizes abstract transformations, embedding them into persistent networks of material and social interactions. In this sense, praxis preserves homological invariants by reproducing relational structures, while simultaneously negotiating the constraints and affordances of the real world. The duality with ingenia is thus maintained: conceptual invention provides new forms and relations, and praxis actualizes them, creating a continuous feedback loop through which reality is both represented, transformed, and materially reproduced.

4 Conclusion

The fourfold framework of imagology, homology, ingenia, and praxis provides an integrated program for understanding the production, transformation, and reproduction of knowledge and reality across both the arts and the sciences. Imagology initiates this program by revealing the invisible structures of representation: it studies the relational images, stereotypes, and discursive constructs through which cultures, institutions, and individuals perceive themselves and others. Homology extends this insight, identifying the invariants and structural relations that persist under transformation, whether in textual traditions, scientific models, or social practices. Together, these first two points establish a categorical perspective in which objects, relations, and transformations can be rigorously analyzed, allowing the study of both persistence and variation across domains.

Ingenia moves beyond observation and structural analysis into the realm of conceptual creation. Drawing on the insights of Poincaré, Weyl, Pauling, and Thom, ingenia encompasses the invention of new forms and processes, whether in theoretical physics, chemistry, mathematics, or the arts. It is the capacity to recognize, manipulate, and generate structural patterns and symmetries, transmuting empirical or experiential data into coherent conceptual architectures. In painting, sculpture, or architecture, as in scientific theory, ingenia operates by producing relational configurations that extend existing structures, creating possibilities that are intelligible and applicable across domains. It is the formalization of creativity itself: a rigorous, disciplined transformation of forms, processes, and meanings.

Praxis complements ingenia by enacting these conceptual structures in the material and operational world. As Galison shows in science, as Wiener in cybernetics, and as Carpo in digital fabrication, praxis stabilizes reality through feedback, instrumentation, and iterative reproduction. Classical and historical examples, from Vitruvius' architecture to Marx's analysis of production, demonstrate that reproduction is always constrained by material, technical, and social conditions, linking conceptual invention to concrete realization. Praxis operationalizes the morphisms generated by ingenia, embedding them in networks of social, technological, and physical interactions, thereby preserving relational invariants while producing tangible effects in the world.

Taken together, these four "arts" constitute a programmatic approach to the philosophy of the arts and sciences. Imagology teaches us to see the invisible, homology teaches us to grasp persistent structures, ingenia teaches us to transform and invent, and praxis teaches us to reproduce and intervene in reality. This framework also preserves and illuminates critical dualities: narcissism and empathy, individual and collective, concept and material, creation and reproduction. It situates knowledge, creativity, and action within a unified categorical vision in which representation, transformation, and realization are mutually informing. In this sense, the program fosters a reflective, interdisciplinary mastery, preparing students to navigate and generate knowledge across artistic and scientific domains while remaining attentive to the structural and ethical conditions that govern both the imagination and the material world.

5 Literature Review

- **Bento de Jesus Caraça, *A Cultura Integral do Indivíduo*** [1] A cultural and educational manifesto delivered as a 1933 conference exploring the collective development of human culture and consciousness. B. J. Caraça argues for an integral culture that unifies intellectual, moral, and social dimensions to awaken communal self-understanding in uncertain times.

- **Bruno Lemaitre, *An Essay on Science and Narcissism*** [2] A reflective study on the role of narcissism in contemporary academic science, showing how ego-driven behaviors and self-promotion can shape research priorities, recognition, and institutional reward structures.

- **Hugo Dyserinck, *Zur Entwicklung der komparatistischen Imagologie*** [3] A reflective study on the role of literary representations in the construction of national identities, showing how culturally mediated images of the Self and the Other shape perceptions, reinforce stereotypes, and inform broader ideological and historical narratives.

- **Henri Poincaré, *Science and Hypothesis*** [6] A classic work on the philosophy of science, emphasizing that scientific theories are creative inventions of form rather than mere collections of empirical facts, and framing mathematical structures as products of human imagination.

- **Hermann Weyl, *Symmetry*** [7] An exploration of symmetry as a unifying principle across mathematics, physics, and aesthetics, providing foundational insight into invariance, transformation, and structural reasoning.

- **Linus Pauling, *The Nature of the Chemical Bond*** [8] A foundational text in chemistry showing how abstract concepts like orbitals and valence bonds shape coherent models of molecular structure from empirical data.

- **René Thom, *Structural Stability and Morphogenesis*** [10] An influential work on mathematical models of form and change, introducing concepts of stability and bifurcation with broad applications in science, biology, and the arts.

- **Peter Galison, *Image and Logic: A Material Culture of Microphysics*** [11] A study of how scientific theories, instruments, and experimental practice co-evolve, showing that material culture and epistemic frameworks are deeply intertwined.

- **Norbert Wiener, *Cybernetics*** [12] A foundational work on feedback, control, and communication in machines and organisms, illustrating recursive structures that influence both theory and practice.

- **Mario Carpo, *The Second Digital Turn*** [13] Analysis of digital fabrication, 3D printing, and parametric design as transformative practices that extend conceptual design into new computational and material domains.

- **Vitruvius, *De Architectura*** [14] A classical treatise linking architectural theory, technique, and material realization, exemplifying an early integration of conceptual and practical domains.

- **Karl Marx, *Capital: A Critique of Political Economy*, Vol. 1** [15] A foundational economic critique analyzing how material production structures shape social relations and the reproduction of labor and capital.

Acknowledgments - A.I.M ackowledges the Institute for Comparative Literature; this article was written within the scope of research developed at the Institute for Comparative Literature, an R&D Unit financed by national funds through FCT – Fundação para a Ciência e a Tecnologia (UID/500/2025).

- N.M.F. acknowledges the Portuguese Foundation for Science and Technology (Fundação para a Ciência e Tecnologia) FCT/MCTES (PIDDAC) through the following Projects: Associate Laboratory ARISELA/P/0112/2020; UIDP/04044/2020; UIDB/04044/2020; PAMI - ROTEIRO/0328/2013 (N° 022158); MATIS (CENTRO-01-0145-FEDER-000014 - 3362); FruitPV; Generative.Thermodynamic; by CDRSP and ESTG from the Polytechnic of Leiria. This work also received full or partial financial support from the Research and Innovation Agenda for the Sustainability of Agriculture, Food and Agro-industry (Notice No. 09/C05-i03/2021), under project PRR-C05-i03-I-000251 (Fruit-PV), supported by the Recovery and Resilience Plan (PRR) and European Funds NextGeneration EU.

Referências

[1] B. de Jesus Caraça, *A Cultura Integral do Indivíduo*, Lisbon, 1933. Available at: `https://www.marxists.org/portugues/caraca/1933/05/25.pdf`

[2] B. Lemaitre, *An Essay on Science and Narcissism: How Do High-Ego Personalities Drive Rese-*

arch in Life Sciences?, Lausanne: Presses polytechniques et universitaires romandes, 2015. ISBN: 9782839918411.

[3] H. Dyserinck, *Zur Entwicklung der komparatistischen Imagologie*, Neohelicon, 1988.

[4] Y. N. Harari, *21 Lessons for the 21st Century*, Penguin, Random House, New York, 2018.

[5] Á. M. Machado and D.-H. Pageaux, *Da literatura comparada à teoria da literatura*, Lisboa, Edições 70, 1988.

[6] H. Poincaré, *Science and Hypothesis*, trans. W. J. G. Mawson, Dover Publications, New York, 1952 (original work published 1902).

[7] H. Weyl, *Symmetry*, Princeton University Press, Princeton, 1952.

[8] L. Pauling, *The Nature of the Chemical Bond and the Structure of Molecules and Crystals*, Cornell University Press, Ithaca, 1939.

[9] G. Siebenmann, “Sprache als Faktor der kulturellen Identität: Der Fall Kataloniens,” in H. Dyserinck and K. U. Syndram (eds.), *Komparatistik und Europaforschung: Perspektiven vergleichender Literatur- und Kulturwissenschaft*, Bonn, Bouvier, 1992, pp. 231–251.

[10] R. Thom, *Structural Stability and Morphogenesis*, trans. D. H. Fowler, W. A. Benjamin, Reading, MA, 1972 (original French edition 1970).

[11] P. Galison, *Image and Logic: A Material Culture of Microphysics*, University of Chicago Press, Chicago, 1997.

[12] N. Wiener, *Cybernetics: Or Control and Communication in the Animal and the Machine*, MIT Press, Cambridge, MA, 1948.

[13] M. Carpo, *The Second Digital Turn: Design Beyond Intelligence*, MIT Press, Cambridge, MA, 2017.

[14] Vitruvius, *De Architectura*, trans. F. Granger, Harvard University Press (Loeb Classical Library), Cambridge, MA, 1931.

[15] K. Marx, *Capital: A Critique of Political Economy*, Vol. 1, trans. B. Fowkes, Penguin Classics, London, 1976 (original work published 1867).

Photo by António Calado

1 Scripta-Ingenia Editorial Policy

Leadership

The editorial leadership of *Scripta-Ingenia* is unified under a single individual who serves simultaneously as Director and Chief Editor. This consolidated governance ensures a consistent vision and maintains the highest standards of editorial integrity and scientific credibility.

Chief Editor's Contributions

Each issue features an editorial contribution from the Chief Editor, providing context, perspective, and scholarly leadership. Additional contributions from the Chief Editor may be included when appropriate to enhance the issue's thematic coherence and intellectual resonance.

Article Contributions

Scripta-Ingenia publishes a blend of invited contributions and unsolicited submissions. Invited pieces are selected for their topical relevance and authors' expertise. Unsolicited manuscripts, submitted by members of the scientific and engineering communities, undergo a rigorous peer-review process to ensure quality and relevance.

Editorial Standards

Every article, whether invited or submitted, is evaluated by the Chief Editor and editorial board according to rigorous standards for scientific rigor, accuracy, and relevance. Special issues or thematic sections are regularly curated to spotlight emerging topics or interdisciplinary challenges.

Types of Content

Each issue contains a diverse range of content types, including original research, review articles, case studies, opinion pieces, and editorials. This variety ensures readers encounter a wide spectrum of scientific discourse, from breakthrough findings to thoughtful reflection.

Publication Calendar

Scripta-Ingenia typically releases its primary issue at the Winter Solstice. In years with multiple issues, additional editions coincide with either the Summer Solstice or one of the Equinoxes, providing a steady cadence of fresh scientific content throughout the year.

Author Support

The editorial team prioritizes open, timely, and constructive communication during the peer-review and publication process, fostering a positive and collaborative experience for authors.

By maintaining a cohesive editorial voice and upholding rigorous standards, *Scripta-Ingenia* aspires to serve as a vibrant platform for scientific dialogue, promoting diverse, high-quality content with both academic depth and broad relevance.