Scripta-Ingenia

Verba volant, scripta manent

World Peace Bell, Christchurch Botanic Gardens, New Zealand.

Palavras, leva-as o vento. A honra já não existe e só aquilo que fica escrito é que tem a força de lei. Este é o estado corrente da nossa sociedade tecnologicamente desenvolvida. Se por um lado isso pode ser considerado um aspecto positivo pois estamos de facto a viver numa sociedade tecnologicamente avançada, por outro lado pode levantar a questão de qual o tipo de ciência que está a ser construída, qual a sua moralidade, honra e profundidade. É neste ambiente de grande desconfiança sobre o que é fazer ciência, sobre qual é o papel do cientista no mundo actual, que surge a revista Scripta-Ingenia.

Sinopse A Scripta-Ingenia assume-se como uma revista de divulgação científica tratando temas da ciência e da tecnologia, cobrindo todas as áreas do saber no domínio das ciências exactas ou aplicadas. Interessa-se ainda por artigos de opinião, sobre tópicos científicos ou não, desde que escritos por autores na área das ciências e da engenharia, e que reflitam as suas opiniões enquanto membros dessa comunidade.

Missão Promover a divulgação, difusão e disseminação do conhecimento científico, técnico e pedagógico.

Objetivos

(1) Disseminar e tornar pública a investigação científica desenvolvida, assim como o conhecimento científico produzido, pela comunidade académica, social e empresarial.

(2) Estimular o debate entre académicos, profissionais, cidadãos comuns e outros interessados.

(3) Permitir que estudantes, professores, especialistas e profissionais divulguem a sua pesquisa e conheçam a pesquisa de colegas.

(4) Incentivar a investigação científica e académica e a divulgação de metodologias científicas.

(5) Constituir-se como meio de comunicação entre especialistas e público em geral.

Público-alvo Comunidade Académica do IPLeiria; Alunos, Docentes e investigadores; outros Profissionais que desenvolvam a sua atividade nas áreas do domínio técnico-científico ou outras afins; Público em geral.

Periodicidade Semestral (Solstício de Verão, Solstício de Inverno), sem prejuízo de se proceder a edições extraordinárias sempre que tal se considere justificado, as quais serão então libertadas pela ocasião dos Equinócios, de Primavera ou Outono.

A Scripta-Ingenia[a] assume-se como uma revista de divulgação científica tratando temas da ciência e da tecnologia, cobrindo todas as áreas do saber no domínio das ciências exactas ou aplicadas. Interessa-se ainda por artigos de opinião, sobre tópicos científicos ou não, desde que escritos por autores na área das ciências e da engenharia, e que reflitam as suas opiniões enquanto membros dessa comunidade. Este é o seu número seis e corresponde ao Solstício de Verão do ano de 2016.

Director and Chief Editor — Nelson Martins-Ferreira
CDRSP-ESTG, IPLeiria

[a] A Scripta-Ingenia é uma revista científica de publicação periódica semestral (sem prejuízo de se poder proceder a edições extraordinárias, sempre que tal se considerar justificado), disponibilizada tanto em formato eletrónico como em suporte de papel, cujo acesso é público e gratuito.

É dinamizada pelo Centro para o Desenvolvimento Rápido e Sustentado do Produto (CDRSP) do Instituto Politécnico de Leiria

As línguas de publicação são, preferencialmente, o português e o inglês, aceitando-se porém também trabalhos em outras línguas sempre que tal se justifique.

Julian Huxley (1887–1975)

> By speech first, but far more by writing, man has been able to put something of himself beyond death. In tradition and in books an integral part of the individual persists, for it can influence the minds and actions of other people in different places and at different times: a row of black marks on a page can move a man to tears, though the bones of him that wrote it are long ago crumbled to dust. In truth, the whole progress of civilization is based upon this power.

Julian Huxley, The Individual in the Animal Kingdom

Qu'est-ce *l'admissibilité* de Nelson Martins-Ferreira?

by Tim Van der Linden

Institut de recherche en mathématique et physique
Université catholique de Louvain
chemin du cyclotron 2 bte L7.01.02
1348 Louvain-la-Neuve, Belgium
tim.vanderlinden@uclouvain.be

Abstract In his Ph.D. thesis [MF08a], Nelson Martins-Ferreira introduced a technical condition (for a certain type of diagram in a category) which he called *admissibility*. His first aim was to efficiently describe internal categorical structures, but the flexibility of the condition allowed him to use it for expressing commutativity conditions as well.

Admissibility allowed us to conveniently describe the so-called *Smith is Huq* condition [MFVdL12, HVdL13] and its close relationship with weighted commutativity [GJU14, MFVdL14]. We were, however, not entirely happy to be using a pure technical definition which at first sight does not seem to have a conceptual meaning. Clearly it should model *commuting* in some sense, but *for which kind of objects?*

The aim of my talk is to explain that admissibility is indeed a commutativity condition, namely for the same objects that answer the question raised in [MFMUVdL16].

Merci aux participants au Séminaire Itinérant de Catégories du 31 octobre 2015.

1 Introduction

L'*admissibilité* est une condition technique, en fait une piece de terminologie, introduite par Nelson Martins-Ferreira dans sa thèse de doctorat [MF08a][1]. Nelson avait besoin d'une manière pour exprimer que dans une certaine situation, une certaine flèche existe. Je donnerai les détails plus tard. Le but de mon exposé est de faire de la publicité pour cette condition technique en donnant des exemples concrets, mais aussi de fournir une explication conceptuelle.

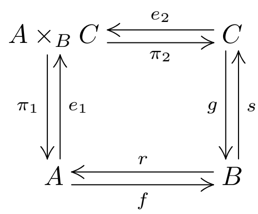
Pourquoi cette condition m'intéresse-t-elle? Elle est d'une flexibilité énorme, qui a permis à Nelson de décrire des structures catégoriques internes (c'était son but dans [MF08a]) tout comme des conditions de commutativité (à la Smith ou à la Huq). Dans notre travail sur les categories internes avec Manfred Hartl [HVdL13], c'était exactement ce qu'il nous fallait pour pouvoir décrire la condition *Smith* = *Huq* en termes de commutateurs de Higgins. Puis, dans [MFVdL14], on s'est aperçu qu'aussi le *commutateur pondéré* de [GJU14] peut être exprimé en utilisant la notion d'admissibilité.

Ce qui nous embêtait en utilisant cette terminologie est le manque d'une explication conceptuelle. Il était clair que d'une certaine manière, l'admissibilité doit modeler *la commutativité*—le fait que deux choses commutent—mais *pour quel genre d'objet?* On était très content donc d'apprendre, en étudiant les idéaux en collaboration avec Nelson Martins-Ferreira, Andrea Montoli et Aldo Ursini dans l'article [MFMUVdL16], qu'en effet une telle explication conceptuelle existe. En fait, la réponse à cette question-ci est exactement la même que celle à la question posée dans [MFMUVdL16].

2 La condition d'admissibilité; les structures catégoriques internes

Dans sa thèse [MF08a], Nelson travaille dans le cadre d'une catégorie qui est pointée, régulière et faiblement de Mal'tsev [MF08b].

Définition 2.1. Une catégorie finiment complète $\mathbb{C}$ est **faiblement de Mal'tsev** si, pour tout produit fibré d'épimorphismes scindés

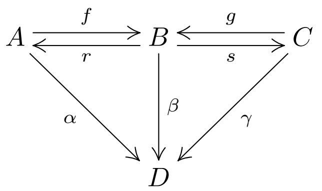


les morphismes $e_1 = \langle 1_A, s \circ f \rangle$ et $e_2 = \langle r \circ g, 1_C \rangle$, induits par la propriété universelle du produit fibré, sont conjointement épimorphes.

Rappelons que $\mathbb{C}$ est une catégorie de Mal'tsev si et seulement si (e_1, e_2) sont conjointement *fortement* épimorphes [Bou96, BB04]. Dans la suite on travaille donc dans une catégorie pointée, régulière et faiblement de Mal'tsev $\mathbb{C}$.

[1] Étudiant de George Janelidze à Aveiro (Portugal). Pendant son travail de doctorat les Janelidze ont déménagé vers l'Afrique du Sud, et il a donc présenté et défendu sa thèse à Cape Town.

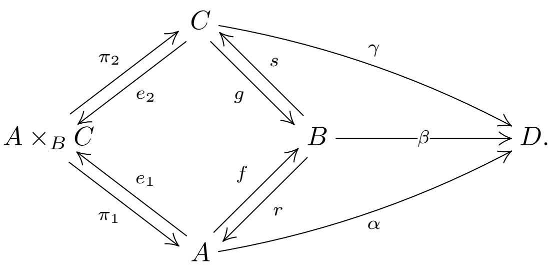
Définition 2.2. Un **diagramme d'admissibilité** (aussi appelé **cerf-volant**, ou **kite** en anglais, par Nelson) est un diagramme de la forme



(2.1)

où $f{\circ}r = 1_B = g{\circ}s$ et $\alpha{\circ}r = \beta = \gamma{\circ}s$.

En prenant le produit fibré de f avec g, on obtient un diagramme



Définition 2.3. Le triple (α, β, γ) est **admissible relatif à** (f, r, g, s) s'il existe un morphisme $\varphi\colon A \times_B C \to D$ tel que $\varphi{\circ}e_1 = \alpha$ and $\varphi{\circ}e_2 = \gamma$. On dit aussi alors que **le cerf-volant** (2.1) **est admissible** (il **vole**).

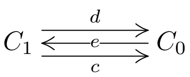
Lemme 2.4. *Un tel morphisme φ (que dans la suite on appellera un **connecteur**) est forcément unique.*

Démonstration. Si aussi $\psi{\circ}e_1 = \alpha$ and $\psi{\circ}e_2 = \gamma$, alors $\psi{\circ}e_1 = \varphi{\circ}e_1$ et $\psi{\circ}e_2 = \varphi{\circ}e_2$. On a donc $\varphi = \psi$ car e_1 et e_2 sont conjointement épimorphes. □

Remarque 2.5. Dans le cadre faiblement de Mal'tsev, l'admissibilité est donc une condition, et non une structure additionnelle, d'un cerf-volant donné.

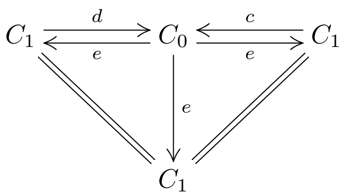
Comme exemple d'application de cette terminologie dans l'étude des structures catégoriques internes, mentionnons le théorème suivant [MF08a] (déjà connu sous une autre forme [CPP92]—voir la prochaine section).

Théorème 2.6. *Un graphe réflexif*



$$d{\circ}e = 1_{C_0} = c{\circ}e$$

est une catégorie interne si et seulement si le cerf-volant



est admissible.

Démonstration. La condition d'admissibilité nous donne une flèche

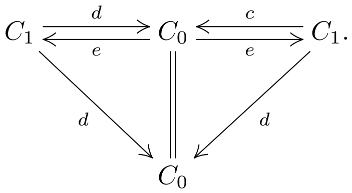
$$m\colon C_1 \times_{C_0} C_1 \to C_1$$

telle que $m{\circ}\langle 1_{C_1}, e{\circ}d\rangle = 1_{C_1}$ et $m{\circ}\langle e{\circ}c, 1_{C_1}\rangle = 1_{C_1}$. Ce m est donc une structure de graphe multiplicatif sur le graphe réflexif donné, qui « envoie une paire

$$(x, y) \qquad \cdot \xleftarrow{\;x\;} \cdot \xleftarrow{\;y\;} \cdot$$

sur sa composée $m(x, y)$ ». Cette multiplication a déjà des unités à gauche et à droite. Il suffit maintenant de montrer que dans une catégorie faiblement de Mal'tsev, tout graphe multiplicatif est une catégorie interne: que la multiplication est automatiquement associative, et que $d{\circ}m = d{\circ}\pi_2$ et $c{\circ}m = c{\circ}\pi_1$.

On montre, par exemple, que $d{\circ}m = d{\circ}\pi_2$ en constatant que les flèches $d{\circ}m$ et $d{\circ}\pi_1$ sont toutes les deux connecteur du cerf-volant



En effet,

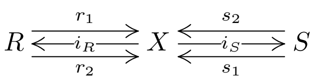
$$\begin{cases} (d{\circ}m){\circ}\langle 1_{C_1}, e{\circ}d\rangle = d = d{\circ}e{\circ}d = (d{\circ}\pi_2){\circ}\langle 1_{C_1}, e{\circ}d\rangle, \\ (d{\circ}m){\circ}\langle e{\circ}c, 1_{C_1}\rangle = d = d{\circ}1_{C_1} = (d{\circ}\pi_2){\circ}\langle e{\circ}c, 1_{C_1}\rangle. \end{cases}$$

Le résultat suit donc du Lemme 2.4. □

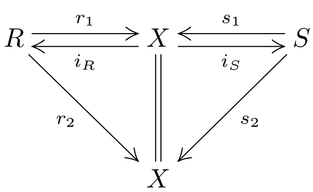
C'est de telle manière que Nelson Martins-Ferreira utilise l'admissibilité dans sa thèse [MF08a].

3 La commutativité pondérée

Bien sûr la Définition 2.3 a été fort inspirée par la notion de connecteur entre deux relations d'équivalence [BG02, BB04], et l'approche à la théorie des commutateurs de Borceux et Bourn en général. En effet, simplement par définition, étant donné deux congruences



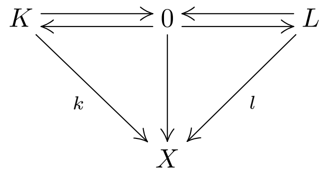
sur un objet X, les relations R et S *commutent dans le sens de Smith* (ou *sont connectées* dans la terminologie de [BB04], voir aussi [Smi76, Ped95]) si et seulement si le diagramme



est un cerf-volant admissible. (En fait, dans [BB04], on inverse les rôles de r_1 et r_2; par symétrie de R on obtient une notion équivalente.) De la même manière, étant donné un cospan

$$K \xrightarrow{k} X \xleftarrow{l} L$$

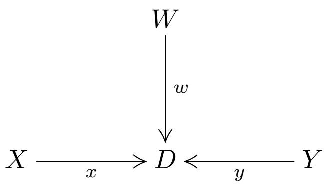
sur on object X, les flèches k et l *commutent dans le sens de Huq* (*collaborent* dans [Bou04], voir aussi [Huq68, BB04]) si et seulement si le diagramme



est un cerf-volant admissible.

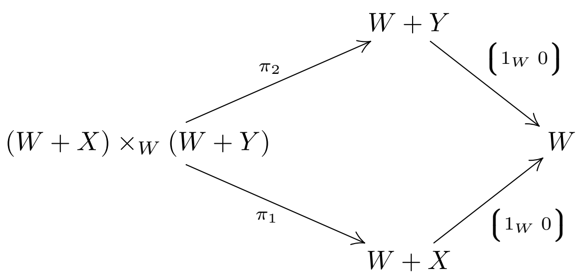
Les deux notions ont été unifiées dans l'article [GJU14] où on introduit le concept de *commutativité pondérée* dans le cadre d'une catégorie homologique finiment cocomplète [BB04].

Définition 3.1. Dans une catégorie homologique finiment cocomplète, un **cospan pondéré** est un triple de morphismes

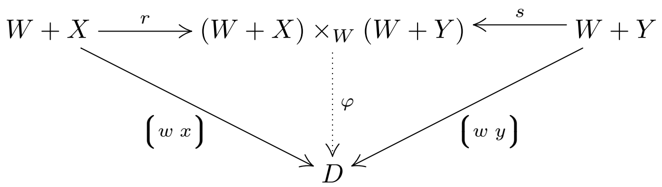


(3.1)

dans lequel (x, y) joue le rôle de cospan et w est le poids. Considérons le produit fibré

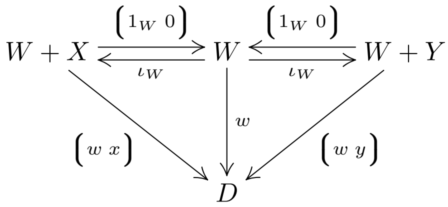


et le diagramme extérieur



induits, $r = \langle 1_{W+X}, \iota_W \circ \begin{pmatrix} 1_W & 0 \end{pmatrix} \rangle$, $s = \langle \iota_W \circ \begin{pmatrix} 1_W & 0 \end{pmatrix}, 1_{W+Y} \rangle$. Dans [GJU14] on dit que les morphismes x et y **commutent au-dessus de** w si et seulement si une flèche φ (appelée une **multiplication interne**) existe telle que le diagramme reste commutatif.

Remarque 3.2. Il est immédiatement clair par définition que les morphismes x et y commutent au-dessus de w si et seulement si le cerf-volant



est admissible. Il est aussi possible (mais un peu plus difficile) de montrer que, réciproquement, l'admissibilité peut être exprimée comme une commutativité pondérée: voir [MFVdL14].

Remarque 3.3. Comme l'explique l'article [GJU14], et tout en accord avec la remarque précédente, en prenant $W = 0$ on retrouve les paires collaborantes (x et y commutent au-dessus de 0 ssi ils commutent dans le sens de Huq), et $w = 1_D$ capte les congruences commutantes dans le sens de Smith (les normalisations respectives x et y de deux congruences R et S sur D commutent au-dessus de 1_D si et seulement si R et S sont connectées, point de vue menant vers le *commutateur d'Ursini* [Man12]).

4 Conditions en termes des commutateurs de Higgins

Restant pour le moment dans le cadre d'une catégorie homologique finiment cocomplète, on donne une analyse de l'admissibilité en termes du commutateur (binaire et ternaire) de Higgins [Hig56, MM10, HL13, HVdL13].

Définition 4.1. Étant donné deux sous-objets $k\colon K \to X$ et $l\colon L \to X$ d'un objet X, leur **commutateur de Higgins** $[K, L] \leq X$ est l'image du morphisme induit

$$K \diamond L \xrightarrow{\iota_{K,L}} K + L \xrightarrow{\begin{pmatrix} k & l \end{pmatrix}} X,$$

où

$$K \diamond L = \operatorname{Ker}\left(\begin{pmatrix} 1_K & 0 \\ 0 & 1_L \end{pmatrix} : K + L \to K \times L \right).$$

Remarque 4.2. Comme expliqué dans [GJU14], le commutateur binaire de Higgins peut être vu comme un commutateur pondéré.

Remarque 4.3. L'objet $K \diamond L$ (et aussi l'objet $K \diamond L \diamond M$ défini en bas) est un exemple d'un *co-smash produit* [CJ03].

Définition 4.4. Si $m\colon M \to X$ est un troisième sous-objet de X, alors le **commutateur ternaire (de Higgins)** $[K, L, M] \leq X$ est défini comme l'image de la flèche composée

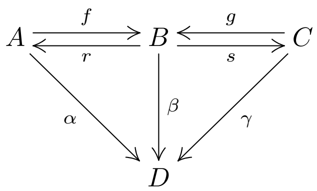
$$K \diamond L \diamond M \xrightarrow{\iota_{K,L,M}} K + L + M \xrightarrow{\begin{pmatrix} k & l & m \end{pmatrix}} X,$$

où $\iota_{K,L,M}$ est le noyau du morphisme

$$K + L + M \xrightarrow{\begin{pmatrix} i_K & i_L & 0 \\ i_K & 0 & i_M \\ 0 & i_L & i_M \end{pmatrix}} (K + L) \times (K + M) \times (L + M).$$

Le théorème suivant [HVdL13] exprime d'admissibilité des cerfs-volants en termes des commutateurs de Higgins.

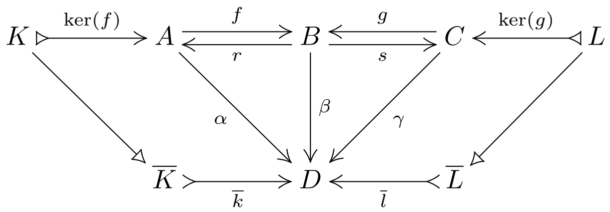
Théorème 4.5. *Le cerf-volant*



est admissible si et seulement si

$$[\overline{K}, \overline{L}] = 0 = [\overline{K}, \overline{L}, \overline{B}]$$

où



$\overline{k}\colon \overline{K} \to D$ *est l'image de* $\alpha \circ \ker(f)$*,* $\overline{l}\colon \overline{L} \to D$ *est l'image de* $\gamma \circ \ker(g)$ *et* $\overline{B}$ *est l'image de* β*.* ☐

Comme une conséquence immédiate du Théorème 2.6, on trouve [HVdL13] une version générale de la caractérisation des catégories internes, valide dans les groupes [Lod82, Mac98] où le commutateur ternaire est invisible, car inclus dans le commutateur binaire:

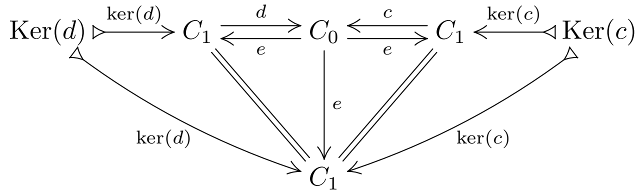
Corollaire 4.6. *Un graphe réflexif*

$$C_1 \underset{c}{\overset{d}{\underset{\leftarrow e \rightarrow}{\rightrightarrows}}} C_0 \qquad\qquad d \circ e = 1_{C_0} = c \circ e$$

est une catégorie interne si et seulement si

$$[\mathrm{Ker}(d), \mathrm{Ker}(c)] = 0 = [\mathrm{Ker}(d), \mathrm{Ker}(c), \mathrm{Im}(e)]$$

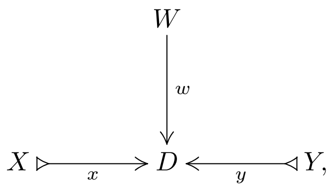
où



et $\mathrm{Im}(e)$ *est l'image de* e*.* ☐

La commutativité pondérée s'exprime de façon suivante [MFVdL14].

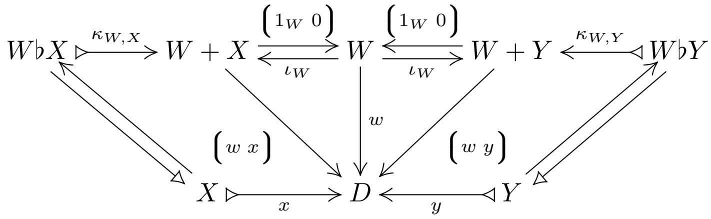
Corollaire 4.7. *Étant donné un cospan pondéré*



où x *et* y *sont des monomorphismes normaux, alors* x *et* y *commutent au-dessus de* w *si et seulement si*

$$[X, Y] = 0 = [X, Y, \mathrm{Im}(w)].$$

Démonstration. Dans le diagramme



le monomorphisme normal x est l'image de $\begin{pmatrix} w & x \end{pmatrix} \circ \kappa_{W,X}$, tandis que y est l'image de $\begin{pmatrix} w & y \end{pmatrix} \circ \kappa_{W,Y}$. En effet, le morphisme $W\flat X \to X$ existe car x est un noyau, et il est scindé par $\eta_X\colon X \to W\flat X$, composante de l'unité de la comonade $W\flat(-)$. ☐

Rappelons [BB04, MFVdL12] qu'une catégorie homologique satisfait la condition **Smith = Huq** si et seulement si deux congruences R, S sur un objet X commutent dans le sens de Smith quand leurs normalisations

$$k = r_2 \circ \ker(r_1)\colon K = \mathrm{Ker}(r_1) \to X$$

et

$$l = s_2 \circ \ker(s_1)\colon L = \mathrm{Ker}(s_1) \to X$$

commutent dans le sens de Huq.

Corollaire 4.8. *La catégorie donnée satisfait* Smith = Huq *si et seulement si pour toute paire de sous-objets normaux* K *et* L *d'un objet* X*,*

$$[K, L] = 0 \qquad \Rightarrow \qquad [K, L, X] = 0.$$

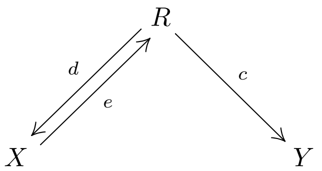
Démonstration. Ceci suit ou bien du corollaire précédent en tenant compte du fait que la commutativité pondérée caractérise les commutativités de Smith et de Huq (voir Remarque 3.3), ou bien directement du Théorème 4.5 en utilisant l'expression de commutativité de Smith ou de Huq en termes d'admissibilité (début de la Section 3). □

5 Les idéaux et les relations scindées surjectives

Selon [JMU07], un **idéal** dans une catégorie pointée régulière est l'image directe d'un noyau le long d'un épimorphisme régulier. Cette définition coincide avec celle proposée dans les variétés [Hig56, Mag67, Urs72].

Dans l'article [MFMUVdL16] on explique qu'un morphisme $i\colon I \to Y$ est un idéal précisément quand il existe une *relation scindée surjective* de laquelle il est la normalisation. Rappelons-nous la définition:

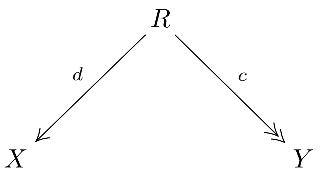
Définition 5.1. Un **span scindé à gauche** de X vers Y est un diagramme



(5.1)

où $d \circ e = 1_X$. Un span scindé à gauche (d, c, e) est appelé une **relation scindée à gauche** quand le span (d, c) conjointement monomorphe.

Un **span surjectif** de X vers Y est un diagramme



où c est un épimorphisme régulier. Un span surjectif (d, c) est une **relation surjective** quand le span (d, c) conjointement monomorphe.

La **normalisation** (la **classe de zéro**) d'un span (d, c) est la flèche

$$c \circ \ker(d)\colon \mathrm{Ker}(d) \to Y.$$

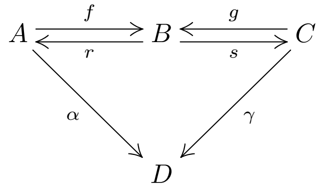
Une flèche $i\colon I \to Y$ est donc un idéal si et seulement si un diagramme comme (5.1) existe tel que d et c sont conjointement monomorphes, c est un épimorphisme régulier et $c \circ \ker(d) = i$.

Retournons maintenant à la condition d'admissibilité dans une catégorie pointée, régulière et faiblement de Mal'tsev. Quand (2.1) est un cerf-volant admissible, alors

$$\alpha \circ r = \varphi \circ e_1 \circ r = \varphi \circ e_2 \circ s = \gamma \circ s,$$

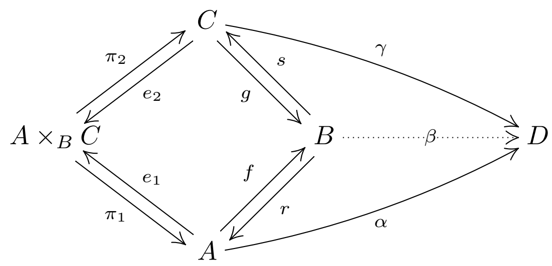
et donc la flèche $\beta = \alpha \circ r = \gamma \circ s$ dans (2.1) est déterminée. Ceci mène naturellement vers la définition suivante, ou le diagramme (2.1) est remplacé par le diagramme plus simple (5.2).

Définition 5.2. On dira que deux spans scindés à gauche (f, α, r) et (g, γ, s) de B vers D comme dans le diagramme



(5.2)

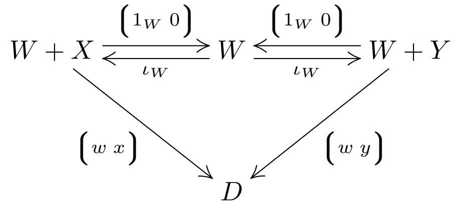
se centralisent ou **commutent** s'il existe un morphisme (forcément unique) $\varphi\colon A \times_B C \to D$, appelé le **connecteur**, du produit fibré $A \times_B C$



de f et g vers l'objet D tel que $\varphi \circ e_1 = \alpha$ et $\varphi \circ e_2 = \gamma$.

Il est clair que cette condition est équivalente à l'admissibilité du cerf-volant (2.1). Ceci explique donc l'admissibilité en termes conceptuels: clairement c'est une condition de commutativité, mais sans la notion de span scindé il était impossible d'exprimer *ce qui commute* exactement.

Remarque 5.3. Bien sûr, cette notion de commutativité de spans scindés est aussi flexible que l'admissibilité. Elle est une extension immédiate de la notion de deux relations d'équivalence connectées dans le sens de [BB04]. Par exemple, dans une catégorie homologique finiment cocomplète, la commutativité pondérée peut être exprimée de manière suivante. Étant donné un cospan pondéré (x, y, w) comme dans (3.1), les morphismes x et y commutent au-dessus de w si et seulement si les spans scindés à gauche $(\begin{pmatrix} 1_W & 0 \end{pmatrix}, \begin{pmatrix} w & x \end{pmatrix}, \iota_W)$ et $(\begin{pmatrix} 1_W & 0 \end{pmatrix}, \begin{pmatrix} w & y \end{pmatrix}, \iota_W)$



commutent.

On finit en mentionnant un théorème de [MFMUVdL16] qui explique pourquoi dans le cadre d'une catégorie semi-abélienne, il était possible de décrire la condition *Smith* = *Huq* en termes d'admissibilité de cerfs-volants—une notion a priori plus générale que la commutativité (dans le sens de Smith) des congruences. En effet, dans une catégorie semi-abélienne, les idéaux et les noyaux coincident; tandis que, par le résultat principal de [BMFVdL13], la condition (i) est exactement *Smith* = *Huq*.

Théorème 5.4. *Dans une catégorie pointée, régulière et faiblement de Mal'tsev* $\mathbb{C}$*, les conditions suivantes sont équivalentes:*

(i) *tout foncteur noyau* $\mathrm{Ker}\colon \mathsf{Pt}_X(\mathbb{C}) \to \mathbb{C}$ *reflète la commutativité des paires de flèches qui ont un idéal comme image;*

(ii) *une paire de relations surjectives scindées à gauche commute dès que ses normalisations commutent;*

(iii) *une paire de spans surjectifs scindés à gauche commute dès que ses normalisations commutent.* □

References

[BB04] F. Borceux and D. Bourn, *Mal'cev, protomodular, homological and semi-abelian categories*, Math. Appl., vol. 566, Kluwer Acad. Publ., 2004.

[BG02] D. Bourn and M. Gran, *Centrality and normality in protomodular categories*, Theory Appl. Categ. **9** (2002), no. 8, 151–165.

[BMFVdL13] D. Bourn, N. Martins-Ferreira, and T. Van der Linden, *A characterisation of the "Smith is Huq" condition in the pointed Mal'tsev setting*, Cah. Topol. Géom. Différ. Catégor. **LIV** (2013), no. 3, 163–183.

[Bou96] D. Bourn, *Mal'cev categories and fibration of pointed objects*, Appl. Categ. Structures **4** (1996), 307–327.

[Bou04] ———, *Commutator theory in regular Mal'cev categories*, Galois Theory, Hopf Algebras, and Semiabelian Categories (G. Janelidze, B. Pareigis, and W. Tholen, eds.), Fields Inst. Commun., vol. 43, Amer. Math. Soc., 2004, pp. 61–76.

[CJ03] A. Carboni and G. Janelidze, *Smash product of pointed objects in lextensive categories*, J. Pure Appl. Algebra **183** (2003), 27–43.

[CPP92] A. Carboni, M. C. Pedicchio, and N. Pirovano, *Internal graphs and internal groupoids in Mal'cev categories*, Am. Math. Soc. for the Canad. Math. Soc., Providence, 1992, pp. 97–109.

[GJU14] M. Gran, G. Janelidze, and A. Ursini, *Weighted commutators in semi-abelian categories*, J. Algebra **397** (2014), 643–665.

[Hig56] P. J. Higgins, *Groups with multiple operators*, Proc. Lond. Math. Soc. (3) **6** (1956), no. 3, 366–416.

[HL13] M. Hartl and B. Loiseau, *On actions and strict actions in homological categories*, Theory Appl. Categ. **27** (2013), no. 15, 347–392.

[Huq68] S. A. Huq, *Commutator, nilpotency and solvability in categories*, Q. J. Math. **19** (1968), no. 2, 363–389.

[HVdL13] M. Hartl and T. Van der Linden, *The ternary commutator obstruction for internal crossed modules*, Adv. Math. **232** (2013), no. 1, 571–607.

[JMU07] G. Janelidze, L. Márki, and A. Ursini, *Ideals and clots in universal algebra and in semi-abelian categories*, J. Algebra **307** (2007), 191–208.

[Lod82] J.-L. Loday, *Spaces with finitely many non-trivial homotopy groups*, J. Pure Appl. Algebra **24** (1982), 179–202.

[Mac98] S. Mac Lane, *Categories for the working mathematician*, second ed., Grad. Texts in Math., vol. 5, Springer, 1998.

[Mag67] R. Magari, *Su una classe equazionale di algebre*, Ann. Mat. Pura Appl. (4) **75** (1967), 277–312.

[Man12] S. Mantovani, *The Ursini commutator as normalized Smith–Pedicchio commutator*, Theory Appl. Categ. **27** (2012), 174–188.

[MF08a] N. Martins-Ferreira, *Low-dimensional internal categorical structures in weakly Mal'cev sesquicategories*, Ph.D. thesis, University of Cape Town, 2008.

[MF08b] ______, *Weakly Mal'cev categories*, Theory Appl. Categ. **21** (2008), no. 6, 91–117.

[MFMUVdL16] N. Martins-Ferreira, A. Montoli, A. Ursini, and T. Van der Linden, *What is an ideal a zero-class of?*, J. Algebra Appl., accepted for publication, 2016.

[MFVdL12] N. Martins-Ferreira and T. Van der Linden, *A note on the "Smith is Huq" condition*, Appl. Categ. Structures **20** (2012), no. 2, 175–187.

[MFVdL14] ______, *A decomposition formula for the weighted commutator*, Appl. Categ. Structures **22** (2014), 899–906.

[MM10] S. Mantovani and G. Metere, *Normalities and commutators*, J. Algebra **324** (2010), no. 9, 2568–2588.

[Ped95] M. C. Pedicchio, *A categorical approach to commutator theory*, J. Algebra **177** (1995), 647–657.

[Smi76] J. D. H. Smith, *Mal'cev varieties*, Lecture Notes in Math., vol. 554, Springer, 1976.

[Urs72] A. Ursini, *Sulle varietà di algebre con una buona teoria degli ideali*, Boll. Unione Mat. Ital. **4** (1972), no. 6, 90–95.

Ficha Técnica
Director: Nelson Martins Ferreira
Proprietário: Instituto Politécnico de Leiria, anotado na ERC
Morada: Rua General Norton de Matos; Apartado 4133; 2411-901 Leiria, Portugal
Director Adjunto: Nuno Alves; Sub Director: Artur Mateus
Editor: Nelson Martins Ferreira; Sede de Redação: Edificio IPL, Rua de Portugal – Zona Industrial 2430-028 – Marinha Grande, Portugal; Contacto: scripta.ingenia@ipleiria.pt
Colaboradores e estatuto editorial: `http://cdrsp.ipleiria.pt/scriptaingenia/`

Internal precategories relative to split epimorphisms

by Nelson Martins-Ferreira

Centre for Rapid and Sustainable Product Development
Polytechnic Institute of Leiria
martins.ferreira@ipleiria.pt

Abstract For a given category B we are interested in studying internal categorical structures in B. This work is the starting point, where we consider reflexive graphs and precategories (i.e., for the purpose of this note, a simplicial object truncated at level 2). We introduce the notions of reflexive graph and precategory relative to split epimorphisms. We study the additive case, where the split epimorphisms are "coproduct projections", and the semi-additive case where split epimorphisms are "semi-direct product projections". The result is a generalization of the well known equivalence between precategories and 2-chain complexes. We also consider an abstract setting, containing, for example, strongly unital categories.

This paper is a revised version of the text [10]. A sequel of its main ideas can be found in [11].
Keywords: Internal category, internal action, half-reflection, crossed-module.
Subject classification [2000]: Primary 18D05, 18D35; Secondary 18E05.

1 Introduction

A (internal) reflexive graph in the category of abelian groups, Ab, is completely determined, up to an isomorphism, by a morphism $h : X \longrightarrow B$ and it can be presented as

$$X \oplus B \underset{[h\,1]}{\overset{\pi_2}{\rightrightarrows}} \!\!\xleftarrow{\iota_2}\! B\ .$$

An internal precategory (i.e., for the purpose of this work, a simplicial object truncated at level 2) is, in the first place, determined by a diagram

$$Y \underset{u}{\overset{a}{\rightrightarrows}} \!\!\xleftarrow{b}\! X \xrightarrow{h} B$$

such that

$$ab = 1 = ub\ ,\ ha = hu,$$

and later, with a further analysis, it simplifies to a 2-chain (see [6] and [3] for more general results on this topic), i.e.,

$$Z \xrightarrow{t} X \xrightarrow{h} B\ \ ,\ \ ht = 0;$$

It can be canonically presented in the form

$$(Z \oplus X) \oplus (X \oplus B) \; \begin{array}{c} \xrightarrow{\pi_2} \\ \xleftarrow{\iota_2} \\ \xrightarrow{[\iota_1[t\,1],1]} \\ \xleftarrow{\iota_2 \oplus \iota_2} \\ \xrightarrow[\pi_2 \oplus [h\,1]]{} \end{array} \; X \oplus B \underset{[h\,1]}{\overset{\pi_2}{\rightrightarrows}} \!\!\xleftarrow{\iota_2}\! B\ .$$

The same result holds true in an arbitrary additive category with kernels. In this work we will be interested in answering the following question: "what is the most general setting where one can still have similar results?".

An old observation of G. Janelidze, says that "since every higher dimensional categorical structure is obtained from an n-simplicial object; and since a simplicial object is build up from split epis; and, since in Ab, every split epi is simply a biproduct projection, then it is expected that, when internal to Ab, all the higher dimensional structures reduce to categories of presheaves". We use this observation as a motivation for the study of internal categorical structures restricted to a given subclass of split epis.

In particular, in this work we will be interested in the study of the notion of internal reflexive graph (1-simplicial object) and internal precategory (2-simplicial object) relative to a given subclass of split epis. The main examples that we have in mind are coproduct projections (in a pointed category with coproducts), product projections, semidirect product projections, etc.

For some cases (and using the words of D. Bourn), the given subclass of split epis is *saturated*. That is the case, for example, in an additive category with kernels when we take the subclass of biproduct projections. In general this is not the case, nevertheless, in some cases, interesting notions do occur.

let us take for example the category of pointed sets and the class of coproduct projections, that is, consider all the split epis of the form

$$X \sqcup B \underset{\iota_2}{\overset{[0\,1]}{\rightleftarrows}} B\ ;$$

It follows that a reflexive graph relative to coproduct projections is completely determined by a morphism

$$h : X \longrightarrow B$$

and it is of the form

$$X \sqcup B \underset{[h\,1]}{\overset{[0\,1]}{\underset{\xleftarrow{\iota_2}}{\rightrightarrows}}} B\ ;$$

A precategory is determined by a diagram

$$Y \underset{u}{\overset{a}{\underset{\xleftarrow{b}}{\rightrightarrows}}} X \xrightarrow{h} B \quad , \quad ab = 1 = ub\ ,\ ha = hu,$$

and it is of the form

$$Y \sqcup (X \sqcup B) \begin{array}{c} \xrightarrow{[0\,1]} \\ \xleftarrow{\iota_2} \\ \xrightarrow{[\iota_1 u,\,1]} \\ \xleftarrow{b \sqcup \iota_2} \\ \xrightarrow{a \sqcup [h\,1]} \end{array} X \sqcup B \underset{[h\,1]}{\overset{[0\,1]}{\underset{\xleftarrow{\iota_2}}{\rightrightarrows}}} B\ .$$

The key factor for this result to hold is the fact that ι_1 is the kernel of $[0\ 1]$.

It turns out that a reflexive graph $(h : X \longrightarrow B)$ can be seen as a precategory, simply by taking

$$X \underset{1}{\overset{1}{\underset{\xleftarrow{1}}{\rightrightarrows}}} X \xrightarrow{h} B\ ,$$

and it is an internal category if the kernel of h is trivial, which is the same as saying that the following square

$$\begin{array}{ccc} X \sqcup (X \sqcup B) & \xrightarrow{[0\,1]} & X \sqcup B \\ {\scriptstyle 1 \sqcup [h\,1]} \downarrow & & \downarrow {\scriptstyle [h\,1]} \\ X \sqcup B & \xrightarrow{[0\,1]} & B \end{array}$$

is a pullback.

Specifically, given a morphism $h : X \longrightarrow B$ with trivial kernel (in pointed sets), the internal category it describes is the following: the objects are the elements of B; the morphisms are the identities 1_b for each $b \in B$ and also the elements $x \in X$, except for the distinguished element $0 \in X$ that is identified with $0 \in B$. The domain of every x in X is $0 \in B$ and the codomain is $h(x)$. Since all arrows (except identities) start from $0 \in B$ and because the kernel of h is trivial, two morphisms x and x' (other than 0) never compose. The picture is a star with all arrows from the origin with no nontrivial loops.

Another example, still in pointed sets, is obtained if considering the subclass of split epis that are product projections, that is, the ones of the form

$$X \times B \underset{<0,1>}{\overset{\pi_2}{\rightleftarrows}} B\ ,$$

then the following result is obtained.

A (internal) reflexive graph relative to product projections is given by a map

$$\xi : X \times B \longrightarrow B; \quad (x, b) \mapsto x \cdot b$$

such that $0 \cdot b = b$ for all $b \in B$; and it is of the form

$$X \times B \underset{\xi}{\overset{\pi_2}{\underset{\xleftarrow{<0,1>}}{\rightrightarrows}}} B\ .$$

A internal precategory, relative to product projections, is given by

$$Y \times (X \times B) \xrightarrow{\mu} X \times B \xrightarrow{\xi} B \quad , \quad Y \underset{\beta}{\overset{\alpha}{\rightleftarrows}} X$$

satisfying

$$\begin{aligned} \alpha\beta &= 1, \\ \mu(y, x, b) &= (y +_b x,\ b)\,, \\ 0 +_b x &= x = \beta(x) +_b 0 \\ (y +_b x) \cdot b &= \alpha(y) \cdot (x \cdot b)\,; \end{aligned}$$

and it is of the form

$$Y \times (X \times B) \begin{array}{c} \xrightarrow{\pi_2} \\ \xleftarrow{<0,1>} \\ \xrightarrow{\mu} \\ \xleftarrow{\beta \times <0,1>} \\ \xrightarrow{\alpha \times \xi} \end{array} X \times B \underset{\xi}{\overset{\pi_2}{\underset{\xleftarrow{<0,1>}}{\rightrightarrows}}} B\ .$$

In particular, if $X = Y$ and $\alpha = \beta = 1$, and the associativity law is satisfied, then we get an internal category. The key point here is the fact that the square

$$\begin{array}{ccc} X \times (X \times B) & \xrightarrow{\pi_2} & X \times B \\ {\scriptstyle 1 \times \xi} \downarrow & & \downarrow {\scriptstyle \xi} \\ X \times B & \xrightarrow{\pi_2} & B \end{array}$$

is a pullback square.

This means that a internal category in pointed sets and relative to product projections is given by two maps

$$\begin{aligned} \mu &: X \times (X \times B) \longrightarrow X \times B; \quad (x, x', b) \mapsto (x +_b x', b) \\ \xi &: X \times B \longrightarrow B; \quad (x, b) \mapsto x \cdot b \end{aligned}$$

such that

$$\begin{aligned} 0 +_b x &= x = x +_b 0 \\ 0 \cdot b &= b \\ (x +_b x') \cdot b &= x \cdot (x' \cdot b) \end{aligned}$$

and, in order to have associativity, we need the additional condition

$$\left(x'' +_{(x \cdot b)} x'\right) +_b x = x'' +_b (x' +_b x)\,.$$

Specifically, given such structure in pointed sets, the corresponding internal category that it represents is described thus: the objects are the elements of B; the morphisms are pairs (x, b) with domain b and codomain $x \cdot b$; the composition of

$$b \xrightarrow{(x,b)} x \cdot b \xrightarrow{(x',b')} x' \cdot (x \cdot b)$$

is the pair $(x' +_b x, b)$.

We will observe that for a given subclass of split epis, when the following two properties are satisfied for every split epi (A, α, β, B) in the subclass:

(a) the morphism $\alpha : A \longrightarrow B$ has a kernel, say $k : X \longrightarrow A$

(b) the pair (k, β) is jointly epic

then a reflexive graph relative to the given subclass of split epis is determined by a split epi in the subclass, say (A, α, β, B) together with a *central morphism*

$$h : X \longrightarrow B.$$

A morphism h is central if there is a (necessarily unique) morphism, denoted by $[h\ 1] : A \longrightarrow B$, such that

$$[h\ 1]\beta = 1 \ \ , \ \ [h\ 1]k = h,$$

with $k : X \longrightarrow A$ the kernel of α (see e.g. [4]).

In the case of Groups, considering the subclass of split epis that are semi-direct product projections,

$$X \rtimes B \underset{<0,1>}{\overset{\pi_2}{\rightleftarrows}} B \ ,$$

the notion of central morphism $h : X \longrightarrow B$ (together with a semidirect product projection, or an internal group action) corresponds to the usual definition of pre-crossed module.

This fact may lead us to consider an abstract notion of semidirect product as a diagram in a category satisfying some universal property.

In [1] O. Berndt proposes the categorical definition of semidirect products as follows: the semidirect product of X and B (in a pointed category) is a diagram

$$X \overset{k}{\longrightarrow} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

such that $\alpha\beta = 1$ and $k = \ker \alpha$.

We observe that it not unreasonable to adjust this definition to the following one: in a pointed category, the semidirect product of X and B, denoted $X \rtimes B$, is defined together with two morphisms

$$X \overset{k}{\longrightarrow} X \rtimes B \overset{\beta}{\longleftarrow} B$$

satisfying the following three conditions:
(a) the pair (k, β) is jointly epic
(b) the zero morphism

$$0 : X \longrightarrow B$$

is central, that is, there exists a (necessarily unique) morphism $[0\ 1] : X \rtimes B \longrightarrow B$ with $[0\ 1]\beta = 1$ and $[0\ 1]k = 0$
(c) k is the kernel of $[0\ 1]$.

We must add that this object $X \rtimes B$ may not be uniquely determined (not even up to an isomorphism), to achieve that fact we should require, in addition that the pair (k, β) is universal in some suitable sense.

We have not investigated further the consequences of such a definition. This will be done in a future work[2] .

We have chosen to mention it here only because it is closely related to this work.

Another example, for considering internal categories relative to split epis, can be found in [12] where A. Patchkoria shows that, in the category of Monoids, the notion of internal category relative to semidirect product projections is in fact equivalent to the notion of a Schreier category.

This work is organized along the following lines.

First we recall some basic definitions, and introduce a concept that is obtained by weakening the notion of reflection, so that we choose to call it half-reflection.

Next we study the case of additivity, and find minimal conditions on a category $\mathbf{B}$ in order to have

$$\begin{array}{rcl} RG\,(\mathbf{B}) & \sim & Mor\,(\mathbf{B}) \\ PC\,(\mathbf{B}) & \sim & 2\text{-}Chains\,(\mathbf{B}) \end{array}$$

the usual equivalences between reflexive graphs and morphisms in $\mathbf{B}$, precategories and 2-chains in $\mathbf{B}$. We show that this is the case exactly when $\mathbf{B}$ is pointed (but not necessarily with a zero object), has binary coproducts and kernels of split epis, and satisfies the following two conditions (see Theorem 9):

(a) ι_1 is the kernel of $[01]$

$$X \overset{\iota_1}{\longrightarrow} X \sqcup B \overset{[0\ 1]}{\longrightarrow} B$$

(b) the split short five lemma holds.

Later we investigate the same notions, and essentially obtain the same results, for the case of semi-additivity, by replacing coproduct projections by semidirect product projections, where the notion of semi-direct product is associated with the notion of internal actions in the sense of [7] and [5].

At the end we describe the same situation for a general setting, specially designed to mimic internal actions and semidirect products. An application of the results is given for the category of unitary magmas with right cancellation.

2 Definitions

Definition 1 (reflection)**.** A functor $I : \mathbf{A} \longrightarrow \mathbf{B}$ is a reflection when there is a functor

$$H : \mathbf{B} \longrightarrow \mathbf{A}$$

and a natural transformation

$$\rho : 1_{\mathbf{A}} \longrightarrow HI$$

[2] As this is a revised version of an older text [10], what was considered as future work at that time is now published work, see [11].

satisfying the following conditions

$$\begin{aligned} IH &= 1_{\mathbf{B}} \\ I \circ \rho &= 1_I \\ \rho \circ H &= 1_H. \end{aligned}$$

Definition 2 (half-reflection)**.** A pair of functors

$$\mathbf{A} \underset{G}{\overset{I}{\rightleftarrows}} \mathbf{B} \quad , \quad IG = 1_{\mathbf{B}}$$

is said to be a half-reflection if there is a natural transformation

$$\pi : 1_{\mathbf{A}} \longrightarrow GI$$

such that

$$I \circ \pi = 1_I.$$

Theorem 1. *For a half-reflection (I, G, π) we always have*

$$\begin{array}{ccc} 1 & \xrightarrow{\pi} & GI \\ & \searrow^{\pi} & \downarrow{\scriptstyle \pi \circ GI} \\ & & GI \end{array} \quad . \tag{2.1}$$

Proof. By naturality of π we have

$$\begin{array}{ccc} A & \xrightarrow{\pi_A} & GIA \\ {\scriptstyle \pi_A}\downarrow & & \downarrow{\scriptstyle GI(\pi_A)} \\ GIA & \xrightarrow{\pi_{GIA}} & GI(GIA) \end{array}$$

but $GI(GIA) = GIA$ and $GI(\pi_A) = 1_{GIA}$. ☐

When it exists, the natural transformation $\pi : 1 \longrightarrow GI$ is *essentially* unique, in the sense that any other such, say $\pi' : 1 \longrightarrow GI$ (with $I \circ \pi' = 1_I$), it is of the form

$$\pi'_A = \pi'_{GIA} \pi_A.$$

The name half-reflection is motivated because if instead of (2.1) we have $\pi \circ G = 1_G$ then the result is a reflection.

For any category $\mathbf{B}$ we consider the category of internal reflexive graphs in $\mathbf{B}$, denoted $RG(\mathbf{B})$ as usual: Objects are diagrams in $\mathbf{B}$ of the form

$$C_1 \overset{d}{\underset{c}{\overset{\longleftarrow e}{\rightrightarrows}}} C_0 \quad , \quad de = 1 = ce;$$

Morphisms are pairs (f_1, f_0) making the obvious squares commutative in the following diagram

$$\begin{array}{ccc} C_1 & \overset{d}{\underset{c}{\overset{\longleftarrow e}{\rightrightarrows}}} & C_0 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_0} \\ C'_1 & \overset{d}{\underset{c}{\overset{\longleftarrow e'}{\rightrightarrows}}} & C'_0 \end{array} .$$

We will also consider the category of internal precategories in $\mathbf{B}$, denoted $PC(\mathbf{B})$, where objects are diagrams of the form

$$C_2 \underset{\pi_1}{\overset{\pi_2}{\substack{\longrightarrow \\ \xleftarrow{e_2} \\ \xrightarrow{m} \\ \xleftarrow{e_1} \\ \longrightarrow}}} C_1 \overset{d}{\underset{c}{\overset{\longleftarrow e}{\rightrightarrows}}} C_0$$

such that

$$\begin{array}{ccc} C_2 & \underset{e_2}{\overset{\pi_2}{\rightleftarrows}} & C_1 \\ {\scriptstyle \pi_1}\downarrow\uparrow{\scriptstyle e_1} & & {\scriptstyle c}\downarrow\uparrow{\scriptstyle e} \\ C_1 & \underset{e}{\overset{d}{\rightleftarrows}} & C_0 \end{array} \tag{2.2}$$

is a split square (i.e. a split epi in the category of split epis), so that in particular we have

$$de = 1_{C_0} = ce \tag{2.3}$$

and furthermore, the following three conditions are satisfied

$$dm = d\pi_2 \tag{2.4}$$

$$cm = c\pi_1. \tag{2.5}$$

$$me_1 = 1_{C_1} = me_2; \tag{2.6}$$

and obvious morphisms.

A precategory in this sense becomes a category[3] if the top and left square in (2.2) is a pullback, and if the associativity condition is satisfied.

Definition 3. A category is said to have coequalizers of reflexive graphs if for every reflexive graph

$$C_1 \overset{d}{\underset{c}{\overset{\longleftarrow e}{\rightrightarrows}}} C_0 \quad , \quad de = 1 = ce$$

the coequalizer of d and c exists.

Definition 4 (pointed category)**.** A pointed category is a category enriched in pointed sets. More specifically, for every pair X, Y of objects, there is a specified morphism, $0_{X,Y} : X \longrightarrow Y$ with the following property:

$$X \overset{0_{X,Y}}{\longrightarrow} Y \overset{f}{\longrightarrow} Z \overset{0_{Z,W}}{\longrightarrow} W$$

$$0_{Z,W} f = 0_{Y,W} \tag{2.7}$$

$$f 0_{X,Y} = 0_{X,Z}. \tag{2.8}$$

Definition 5 (additive category)**.** An additive category is an *Ab*-category with binary biproducts.

Observe that on the contrary to the usual practice we are not considering the existence of a null object, neither in pointed nor in additive categories.

[3]The term precategory is often used when (2.6) is not present; we also observe that in many interesting cases (for example in Mal'cev categories) assuming only (2.6) and the fact that (2.2) is a pullback, then the resulting structure is already an internal category, because associativity comes for free.

3 Additivity

Let $\mathbf{B}$ be any category and consider the pair of functors

$$\mathbf{B}\times\mathbf{B} \underset{G}{\overset{I}{\rightleftarrows}} \mathbf{B}$$

with $I(X,B)=B$ and $G(B)=(B,B)$.

Theorem 2. *The above pair (I,G) is a half-reflection if and only if the category $\mathbf{B}$ is pointed.*

Proof. If $\mathbf{B}$ is pointed simply define

$$\pi_{(X,B)}:(X,B)\longrightarrow(B,B)$$

as $\pi_{(X,B)}=(0_{X,B},1_B)$.
Now suppose there is a natural transformation

$$\pi:1_{\mathbf{B}\times\mathbf{B}}\longrightarrow GI,$$

such that $I\circ\pi=1_I$, this is the same as having for every pair X,B in $\mathbf{B}$ a specified morphism

$$\pi_{X,B}:X\longrightarrow B$$

and conditions (2.7) and (2.8) follow by naturality:

$$\begin{array}{ccc} (Y,W) & \xrightarrow{(\pi_{Y,W},1)} & (W,W) \\ \downarrow{\scriptstyle (f,1)} & & \downarrow{\scriptstyle (1,1)} \\ (Z,W) & \xrightarrow{(\pi_{Z,W},1)} & (W,W) \end{array}\quad,\qquad \begin{array}{ccc} (X,Y) & \xrightarrow{(\pi_{X,Y},1)} & (Y,Y) \\ \downarrow{\scriptstyle (1,f)} & & \downarrow{\scriptstyle (f,f)} \\ (X,Z) & \xrightarrow{(\pi_{X,Z},1)} & (Z,Z) \end{array}\quad.$$

□

Theorem 3. *The functor G as above admits a left adjoint if and only if the category $\mathbf{B}$ has binary coproducts.*

Proof. As it is well known, coproducts are obtained as the left adjoint to the diagonal functor. □

Theorem 4. *Given a half-reflection (I,G,π)*

$$\mathbf{A} \underset{G}{\overset{I}{\rightleftarrows}} \mathbf{B}\quad,\quad \pi:1_{\mathbf{A}}\longrightarrow GI,$$

if the functor G admits a left adjoint

$$(F,G,\eta,\varepsilon),$$

then, there is a canonical functor

$$\mathbf{A}\longrightarrow Pt(\mathbf{B})$$

sending an object $A\in\mathbf{A}$ to the split epi

$$FA \underset{I(\eta_A)}{\overset{\varepsilon_{IA}F(\pi_A)}{\rightleftarrows}} IA\;.$$

Proof. We only have to prove

$$\varepsilon_{IA}F(\pi_A)\,I(\eta_A)=1_{IA}.$$

Start with

$$\pi_A=G(\varepsilon_{IA}F(\pi_A))\,\eta_A$$

and apply I to both sides to obtain

$$I(\pi_A)=\varepsilon_{IA}F(\pi_A)\,I(\eta_A),$$

by definition we have $I(\pi_A)=1_{IA}$. □

In particular if $\mathbf{B}$ is pointed and has binary coproducts we have the canonical functor

$$\mathbf{B}\times\mathbf{B}\xrightarrow{T}Pt(\mathbf{B})$$

sending a pair (X,B) to the split epi

$$X\sqcup B \underset{\iota_2}{\overset{[0\,1]}{\rightleftarrows}} B\;.$$

Theorem 5. *Let $\mathbf{B}$ be a pointed category with binary coproducts. The canonical functor $\mathbf{B}\times\mathbf{B}\xrightarrow{T}Pt(\mathbf{B})$ admits a right adjoint, S, such that $IS=I'$*

$$\begin{array}{ccc} \mathbf{B}\times\mathbf{B} & \xleftarrow{\;S\;} & Pt(\mathbf{B}) \\ & {\scriptstyle I}\searrow \quad \swarrow{\scriptstyle I'} & \\ & \mathbf{B} & \end{array}$$

if and only if the category $\mathbf{B}$ has kernels of split epis.
The functor I' sends a split epi (A,α,β,B) to B.

Proof. If the category has kernels of split epis, then for every split epi we choose a specified kernel

$$X\xrightarrow{k}A\underset{\beta}{\overset{\alpha}{\rightleftarrows}}B$$

and the functor S, sending (A,α,β,B) to the pair (X,B) is the right adjoint for T:

$$\begin{array}{c} (Y,D) \\ \downarrow{\scriptstyle (f,g)} \\ (X,B) \end{array}\qquad \begin{array}{ccccc} Y & \xrightarrow{\iota_1} & Y\sqcup D & \underset{\iota_2}{\overset{[0\,1]}{\rightleftarrows}} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle [kf\,\beta g]} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{k} & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}\;.$$

Now suppose S is a right adjoint to T and it is such that a split epi (A,α,β,B) goes to a pair of the form

$$(K[\alpha],B)$$

with unit and counit as follows

$$\begin{array}{c} (X,B) \\ \downarrow{\scriptstyle (\eta_X,1)} \\ (K[0\,1],B) \end{array}\qquad \begin{array}{ccccc} K[\alpha] & \xrightarrow{\iota_1} & K[\alpha]\sqcup B & \underset{\iota_2}{\overset{[0\,1]}{\rightleftarrows}} & B \\ & {\scriptstyle \varepsilon_1}\searrow & \downarrow{\scriptstyle [\varepsilon_1\,\beta]} & & \downarrow{\scriptstyle 1} \\ & & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}\;.$$

We have to prove that $\varepsilon_1 = \ker \alpha$, and in fact, we have $\alpha\varepsilon_1 = 0$ and by the universal property of the counit we have that given a morphism of split epis

$$\begin{array}{ccccc} X & \xrightarrow{\iota_1} & X \sqcup B & \underset{\iota_2}{\overset{[0\,1]}{\rightleftarrows}} & B \\ & \searrow^{f} & \downarrow{\scriptstyle [f\,\beta]} & & \downarrow{\scriptstyle 1} \\ & & A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \end{array}$$

that is a morphism $f : X \longrightarrow A$ such that $\alpha f = 0$, there exists a unique $(f', 1) : (X, B) \longrightarrow (K[\alpha], B)$ such that

$$[\varepsilon_1\ \beta]\,(f' \sqcup 1) = [f\ \beta]$$

which is equivalent to say $\varepsilon_1 f' = f$. Hence ε_1 is a kernel for α. □

Theorem 6. *Let* $\mathbf{B}$ *be a pointed category with binary coproducts and kernels of split epis. If the canonical adjunction*

$$\mathbf{B} \times \mathbf{B} \underset{S}{\overset{T}{\rightleftarrows}} Pt\,(\mathbf{B}) \quad (T \dashv S)$$

is an equivalence, then:

$$RG\,(\mathbf{B}) \sim Mor\,(\mathbf{B})$$

and

$$PC\,(\mathbf{B}) \sim 2\text{-}Chains\,(\mathbf{B})\,.$$

Proof. By the equivalence we have that a split epi

$$C_1 \underset{e}{\overset{d}{\rightleftarrows}} C_0 \qquad , \quad de = 1$$

is of the form

$$X \sqcup B \underset{\iota_2}{\overset{[0\,1]}{\rightleftarrows}} B\,,$$

and to give a morphism $c : X \sqcup B \longrightarrow B$ such that $c\iota_2 = 1$ is to give a morphism

$$h : X \longrightarrow B.$$

So that a reflexive graph is, up to isomorphism, of the form

$$X \sqcup B \ \overset{[0\,1]}{\underset{[h\,1]}{\substack{\longrightarrow \\ \xleftarrow{\iota_2} \\ \longrightarrow}}}\ B\,.$$

To investigate a precategory we observe that the square (2.2) may be considered as a split epi in the category of split epis, and hence, it is given up to an isomorphism in the form

$$\begin{array}{ccc} Y \sqcup (X \sqcup B) & \underset{\iota_2}{\overset{[0\,1]}{\rightleftarrows}} & X \sqcup B \\ {\scriptstyle a\sqcup[h,1]}\downarrow\uparrow{\scriptstyle b\sqcup\iota_2} & & {\scriptstyle [h\,1]}\downarrow\uparrow{\scriptstyle \iota_2} \\ X \sqcup B & \underset{\iota_2}{\overset{[0\,1]}{\rightleftarrows}} & B \end{array} \quad , \quad ab = 1. \tag{3.1}$$

It follows that m, satisfying $m\iota_2 = 1$ is of the form

$$Y \sqcup (X \sqcup B) \overset{[v\ 1]}{\longrightarrow} (X \sqcup B)$$

and hence to give m is to give $v : Y \longrightarrow X \sqcup B$.
Since we also have (2.4) then $[0\ 1]v = 0$, and v factors through the kernel of $[0\ 1]$ which is (see Theorem 9)

$$X \xrightarrow{\iota_1} X \sqcup B \underset{\iota_2}{\overset{[0\,1]}{\rightleftarrows}} B\,.$$

This shows that to give m is to give a morphism

$$u : Y \longrightarrow X$$

and hence m is given as

$$m = [\iota_1 u\ 1] : Y \sqcup (X \sqcup B) \longrightarrow (X \sqcup B)\,.$$

Finally we have that condition (2.5) is equivalent to $ha = hu$ and $m\,(b \sqcup \iota_2)$ is equivalent to $ub = 1$.
Conclusion 1: A precategory in $\mathbf{B}$ is completely determined by a diagram

$$Y \ \overset{a}{\underset{u}{\substack{\longrightarrow \\ \xleftarrow{b} \\ \longrightarrow}}}\ X \xrightarrow{h} B$$

such that

$$ab = 1 = ub\,,\ ha = hu.$$

Continuing with a further analysis we observe that the resulting diagram is in particular a reflexive graph and hence it is, up to isomorphism, of the form

$$Z \sqcup X \ \overset{[0\,1]}{\underset{[t\,1]}{\substack{\longrightarrow \\ \xleftarrow{\iota_2} \\ \longrightarrow}}}\ X \xrightarrow{h} B$$

where $h[0\ 1] = h[t\ 1]$ is equivalent to $ht = 0$.
Conclusion 2: A precategory in $\mathbf{B}$ is completely determined by a 2-chain complex

$$Y \xrightarrow{t} X \xrightarrow{h} B\ \ ,\ \ ht = 0.$$

□

Remark 1. In the future we will not assume the canonical functor T to be an equivalence, and hence the second conclusion will no longer be possible. However, we will be interested in the study of precategories such that (2.2) is of the form (3.1) and in that case, provided that ι_1 is the kernel of $[0\ 1]$ we still can deduce conclusion 1. Such an example is the category of pointed sets: see Introduction.

There is a canonical inclusion of reflexive graphs into precategories, by sending $h : X \longrightarrow B$ to

$$X \ \overset{1}{\underset{1}{\substack{\longrightarrow \\ \xleftarrow{1} \\ \longrightarrow}}}\ X \xrightarrow{h} B\,.$$

Theorem 7. *If $\mathbf{B}$ has coequalizers of reflexive graphs, then the canonical functor*

$$PC(\mathbf{B}) \underset{V}{\longleftarrow} RG(\mathbf{B})$$

has a left adjoint.

Proof. The left adjoint is the following.
Given the precategory

$$\begin{array}{ccccc} Y & \overset{a}{\underset{u}{\underset{\longleftarrow b}{\rightrightarrows}}} & X & \overset{h}{\longrightarrow} & B \\ & & & \underset{\sigma=coeq}{\searrow} & \uparrow h' \\ & & & & X' \end{array}$$

construct the coequalizer of u and a, say σ, and consider the reflexive graph in $\mathbf{B}$ determined by

$$h' : X' \longrightarrow B.$$

This defines a reflection

$$PC(\mathbf{B}) \overset{U}{\longrightarrow} RG(\mathbf{B})$$

with unit

$$\begin{array}{ccccc} Y & \overset{a}{\underset{u}{\underset{\longleftarrow b}{\rightrightarrows}}} & X & \overset{h}{\longrightarrow} & B \\ \sigma u=\sigma a \downarrow & & \sigma \downarrow & & \| \\ X' & \overset{1}{\underset{1}{\underset{\longleftarrow 1}{\rightrightarrows}}} & X' & \overset{h'}{\longrightarrow} & B \end{array} .$$

□

Next we characterize a category $\mathbf{B}$, pointed, with binary coproducts and such that the canonical functor

$$\mathbf{B} \times \mathbf{B} \longrightarrow Pt(\mathbf{B})$$

is an equivalence.

First observe that:

Theorem 8. *If $\mathbf{B}$, as above, also has binary products, then it is an additive category (with kernels of split epis).*

Proof. We simply observe that in particular

$$\begin{array}{ccc} X \sqcup B & \overset{[0\ 1]}{\underset{\iota_2}{\rightleftarrows}} & B \\ \cong \downarrow & & \| \\ X \times B & \overset{\pi_2}{\underset{<0,1>}{\rightleftarrows}} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} X \sqcup B & \overset{[0\ 1]}{\underset{\iota_2}{\rightleftarrows}} & B \\ \cong \downarrow & & \| \\ X \times B & \overset{\pi_2}{\underset{<1,1>}{\rightleftarrows}} & B \end{array} ,$$

since $X \overset{<1,0>}{\longrightarrow} X \times B$ is a kernel for π_2. See [8] for more details. □

Theorem 9. *Let $\mathbf{B}$ be pointed with binary coproducts and kernels of split epis. The following conditions are equivalent:*

(a) *the canonical adjunction*

$$\mathbf{B} \times \mathbf{B} \underset{S}{\overset{T}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} Pt(\mathbf{B})$$

$$\begin{aligned} T(X,B) &= (X \sqcup B, [0\ 1], \iota_2, B) \\ S(A,\alpha,\beta,B) &= (K[\alpha], B), \end{aligned}$$

is an equivalence of categories;

(b) *the category $\mathbf{B}$ satisfies the following two axioms:*

(A1) *for every diagram of the form*

$$X \overset{\iota_1}{\longrightarrow} X \sqcup B \overset{[0\ 1]}{\underset{\iota_2}{\rightleftarrows}} B \qquad (3.2)$$

the morphism ι_1 is the kernel of $[0\ 1]$;

(A2) *the split short five lemma holds, that is, given any diagram of split epis and respective kernels*

$$\begin{array}{ccccc} X & \overset{k}{\longrightarrow} & A & \overset{\alpha}{\underset{\beta}{\rightleftarrows}} & B \\ f\downarrow & & h\downarrow & & \downarrow g \\ X' & \overset{k'}{\longrightarrow} & A' & \overset{\alpha'}{\underset{\beta'}{\rightleftarrows}} & B' \end{array} \qquad (3.3)$$

if g and f are isomorphisms then h is an isomorphism.

Proof. $(b) \Rightarrow (a)$ Using (A1) we have that $ST \cong 1$, and using (A2) we have, in particular, that $[k\ \beta]$ as in

$$\begin{array}{ccccc} X & \overset{\iota_1}{\longrightarrow} & X \sqcup B & \overset{[0\ 1]}{\underset{\iota_2}{\rightleftarrows}} & B \\ \| & & [k\ \beta]\downarrow & & \| \\ X & \overset{k}{\longrightarrow} & A & \overset{\alpha}{\underset{\beta}{\rightleftarrows}} & B \end{array}$$

is an isomorphism, and hence $TS \cong 1$.

$(a) \Rightarrow (b)$ Suppose $ST \cong 1$, this gives (A1); suppose $TS \cong 1$, so that from (3.3) we can form

$$\begin{array}{ccccc} X & \overset{\iota_1}{\longrightarrow} & X \sqcup B & \overset{[0\ 1]}{\underset{\iota_2}{\rightleftarrows}} & B \\ f\downarrow & & f\sqcup g\downarrow & & \downarrow g \\ X' & \overset{\iota_1}{\longrightarrow} & X \sqcup B & \overset{[0\ 1]}{\underset{\iota_2}{\rightleftarrows}} & B \end{array}$$

and if f, g are isomorphisms, we can find $h^{-1} = [k\ \beta]\left(f^{-1} \sqcup g^{-1}\right)[k'\ \beta']^{-1}$. □

Corollary 1. If T is a reflection then it is an equivalence of categories.

We may now state the following results.

Conclusion 1. Let $\mathbf{B}$ be a pointed category with binary coproducts. TFAE:

(a) T is a reflection and $\mathbf{B}$ has binary products;

(b) $\mathbf{B}$ is additive and has kernels of split epis.

Conclusion 2. Let $\mathbf{B}$ be pointed, with binary products and coproducts and kernels of split epis. TFAE:

(a) T is a reflection;

(b) $\mathbf{B}$ is additive.

3.1 Restriction to split epis

Suppose now that the canonical functor T is not an equivalence, but we still have axiom (3.2), that is $ST \cong 1$. The results relating precategories and reflexive graphs will still hold if we restrict $PC(\mathbf{B})$ to diagrams of the form

$$Y \sqcup (X \sqcup B) \underset{a \sqcup c}{\overset{[0\,1]}{\substack{\longrightarrow \\ \xleftarrow{\iota_2} \\ \xrightarrow{m} \\ \xleftarrow{b \sqcup c} \\ \longrightarrow}}} X \sqcup B \underset{c}{\overset{[0\,1]}{\substack{\longrightarrow \\ \xleftarrow{\iota_2} \\ \longrightarrow}}} B \; .$$

This result will be proved in a more general case in the next sections.

An example of such a case is the category of pointed sets.

If starting with a general half-reflection

$$\mathbf{A} \underset{G}{\overset{I}{\rightleftarrows}} \mathbf{B} \quad , \quad \pi : 1 \longrightarrow GI$$

such that G admits a left adjoint

$$(F, G, \eta, \varepsilon)$$

we consider the canonical functor

$$\mathbf{A} \xrightarrow{T} Pt(\mathbf{B})$$

and ask if it is an equivalence; if not we then ask if it satisfies at least one of the axioms (3.2) or (3.3). For example for $\mathbf{A} = \mathbf{B} \times \mathbf{B}$ and assuming the constructions as above, in the case of pointed sets we have (3.2) but not (3.3), while in groups we have (3.3) but not (3.2).

In the case where we have only (3.2) we will be interested in the study of $RG(\mathbf{B})$ and $PC(\mathbf{B})$ restricted to split epis of the form

$$FA \underset{I(\eta_A)}{\overset{\varepsilon_{IA} F(\pi_A)}{\rightleftarrows}} IA \; ,$$

while if in the presence of (3.3), but not (3.2), we may construct a category of internal actions as suggested in [7].

4 Semi-Additivity

Let $\mathbf{B}$ be a pointed category with binary coproducts and kernels of split epis. As shown in the previous section there is a canonical adjunction

$$\mathbf{B} \times \mathbf{B} \underset{S}{\overset{T}{\rightleftarrows}} Pt(\mathbf{B}) \, , \quad (T \dashv S) \qquad (4.1)$$

We are considering $\mathbf{B} \times \mathbf{B}$ and $Pt(\mathbf{B})$ as objects in the category of functors over $\mathbf{B}$, that is

$$\begin{array}{ccc} \mathbf{B} \times \mathbf{B} & & Pt(\mathbf{B}) \\ & {\scriptstyle I} \searrow \quad \swarrow {\scriptstyle I'} & \\ & \mathbf{B} & \end{array}$$

where $I(X, B) = B$ and $I'(A, \alpha, \beta, B) = B$.

We are also interested in the fact that I is a half-reflection, with respect to some functor G. In the case of $\mathbf{B} \times \mathbf{B}$ there is a canonical choice for G, namely the diagonal functor, and it is a half-reflection if and only if $\mathbf{B}$ is pointed. We are also interested in the fact that G admits a left adjoint.

In the case of $Pt(\mathbf{B})$ there are apparently many good choices for the functor G' to be a half-reflection together with I'. Nevertheless, if we ask that the left adjoint for G' to be F', such that $F'(A, \alpha, \beta, B) = A$, then we calculate G' as follows.

Theorem 10. *Let $\mathbf{B}$ be a category and consider the two functors*

$$Pt(\mathbf{B}) \underset{F}{\overset{I}{\rightrightarrows}} \mathbf{B}$$

$$\begin{aligned} I(A, \alpha, \beta, B) &= B \\ F(A, \alpha, \beta, B) &= A. \end{aligned}$$

The functor F admits a right adjoint

$$(F, G, \eta, \varepsilon)$$

such that $IG = 1_{\mathbf{B}}$ if and only if the category $\mathbf{B}$ has an endofunctor

$$G_1 : \mathbf{B} \longrightarrow \mathbf{B}$$

and natural transformations

$$G_1(B) \underset{\varepsilon_B}{\overset{\pi_B}{\substack{\longrightarrow \\ \xleftarrow{\delta_B} \\ \longrightarrow}}} B \quad , \quad \pi_B \delta_B = 1_B,$$

satisfying the following property:
for every diagram in $\mathbf{B}$ of the form

$$\begin{array}{ccc} A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B' \\ & {\scriptstyle f} \searrow & \\ & & B \end{array} \quad , \quad \alpha\beta = 1$$

there exists a unique morphism

$$f' : A \longrightarrow G_1(B)$$

such that

$$\begin{aligned} \varepsilon_B f' &= f \\ \delta_B \pi_B f' &= f' \beta\alpha. \end{aligned}$$

Proof. Suppose we have $G_1, \pi, \delta, \varepsilon$ satisfying the required conditions in the Theorem, then the functor

$$G(B) = \; G_1(B) \underset{\delta_B}{\overset{\pi_B}{\rightleftarrows}} B$$

is a right adjoint to F; in fact (see [9], p.83, Theorem 2, (iii)) we have functors F and G, and a natural transformation $\varepsilon : FG \longrightarrow 1$, such that each $\varepsilon_B : FG(B) \longrightarrow B$ is universal from F to B:

$$\begin{array}{ccc} A & & A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \\ \downarrow f & : & \downarrow f_1 \quad \searrow f \quad \downarrow f_0 \\ B' & & G_1(B') \underset{\delta_B}{\overset{\pi_B}{\rightleftarrows}} B' \end{array}$$

given f, there is a unique f_1 (with $\varepsilon_B f' = f$, $\delta_B \pi_B f' = f'\beta\alpha$) and f_0 follows as $f_0 = \pi_B f_1 \beta$; conversely, given f_1, we find $f = \varepsilon_B f_1$.

Now, given an adjunction

$$(F, G, \eta, \varepsilon)$$

such that $IG = 1$, if writing

$$G(B) = \; G_1(B) \underset{G_3(B)}{\overset{G_2(B)}{\rightleftarrows}} B$$

we define

$$G_1 = FG \;\; , \;\; \pi_B = G_2(B) \;\; , \;\; \delta_B = G_3(B)$$

and

$$\varepsilon_B : FG(B) \longrightarrow B$$

is the counit of the adjunction.

Clearly we have natural transformations with $\pi_B \delta_B = 1$.

It remains to check the stated property - but it is simply the universal property of ε_B: given a diagram

$$\begin{array}{ll} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B' & , \quad \alpha\beta = 1 \\ \quad \searrow f & \\ \qquad B & \end{array}$$

there is a unique morphism of split epis

$$\begin{array}{ccc} A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B \\ \downarrow f_1 & & \downarrow f_0 \\ G_1(B') & \underset{\delta_B}{\overset{\pi_B}{\rightleftarrows}} & B' \end{array}$$

such that $\varepsilon_B f_1 = f$; being a morphism of split epis means that $f_0 = \pi_B f_1 \beta$, and f_1 is such that $\delta_B \pi_B f_1 = f_1 \beta\alpha$. □

If $\mathbf{B}$ has binary products, then for every $B \in \mathbf{B}$,

$$B \times B \;\underset{\pi_1}{\overset{\pi_2}{\rightrightarrows}}\; B, \qquad <1,1> : B \longrightarrow B \times B$$

satisfies the required conditions and hence F has a right adjoint, G, sending the object B to the split epi

$$B \times B \underset{<1,1>}{\overset{\pi_2}{\rightleftarrows}} B\;.$$

And furthermore, in this case the pair (I, G) is a half-reflection. For the general case, if we ask that (I, G) is a half-reflection, then the following result suffices.

Corollary 2. Let $\mathbf{B}$ be a category and $I, F : Pt(\mathbf{B}) \longrightarrow \mathbf{B}$ as above. If the category $\mathbf{B}$ is equipped with an endofunctor $G_1 : \mathbf{B} \longrightarrow \mathbf{B}$ and natural transformations

$$G_1(B) \;\overset{\pi_B}{\underset{\varepsilon_B}{\rightrightarrows}}\; B, \;\; \delta_B : B \longrightarrow G_1(B) \quad , \quad \pi_B \delta_B = 1_B = \varepsilon_B \delta_B,$$

satisfying the following property:
for every diagram in $\mathbf{B}$ of the form

$$\begin{array}{ll} A \overset{t}{\longleftarrow} A & , \quad t^2 = t \\ \quad \searrow f & \\ \qquad B & \end{array}$$

there exists a unique morphism

$$f' : A \longrightarrow G_1(B)$$

such that

$$\pi_B f' = ft \;,\; \varepsilon_B f' = f \;,\; f't = \delta_B ft,$$

then the functor F has a right adjoint, say G, and the pair (I, G) is a half reflection.

Proof. It is clear that the property is sufficient to obtain G as a right adjoint to F as in the previous Theorem, simply considering $t = \beta\alpha$ and observing that the two conditions $\pi_B f' = ft$, $f't = \delta_B ft$ give $\delta_B \pi_B f' = f'\beta\alpha$: start with $\pi_B f' = ft$, precompose with δ_B, and replace $\delta_B ft$ by $f't$.

As a consequence we have that

$$\pi_B f'\beta = \varepsilon_B f'\beta,$$

since

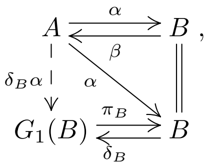
$$\begin{aligned} ft = ftt = \pi_B f't = \pi_B f'\beta\alpha = \pi_B f'\beta \\ ft = \varepsilon_B \delta_B ft = \varepsilon_B f't = \varepsilon_B f'\beta\alpha = \varepsilon_B f'\beta \end{aligned}$$

and hence, given $f : A \longrightarrow B$, we have (f_1, f_0), with $f_1 = f'$ given by the *universal* property and $f_0 = f\beta$ $(= \pi_B f'\beta = \varepsilon_B f'\beta)$.

The pair (I, G) is a half-reflection with

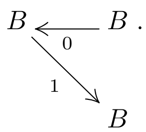
$$\pi : 1_{Pt(\mathbf{B})} \longrightarrow GI$$

given by



and furthermore this is the only possibility. □

Corollary 3. In the conditions of the above Corollary (and assuming $\mathbf{B}$ is pointed), the kernel of π_B is the morphism induced by the diagram



As mentioned in the previous section, if the canonical adjunction (4.1) is not an equivalence we are interested in considering bigger categories, $\mathbf{A}$, that we will call categories of actions, in the place of $\mathbf{B} \times \mathbf{B}$, in order to obtain an equivalence of categories $\mathbf{A} \sim Pt(\mathbf{B})$.

We now turn our attention to the category of internal actions in $\mathbf{B}$.

To define the category of internal actions in $\mathbf{B}$, in the sense of [7], we only need to assume $\mathbf{B}$ to be pointed, with binary coproducts and kernels of split epis: exactly the same conditions necessary to consider the canonical adjunction (4.1); and the construction of the category of internal actions is actually suggested by the adjunction. This seems to suggest an iterative process to obtain bigger and bigger categories "of actions", $\mathbf{A}_1$, $\mathbf{A}_2$, ...

4.1 The category of internal actions

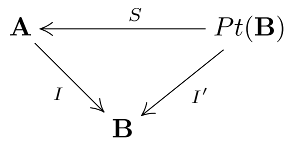
Let $\mathbf{B}$ be a pointed category with binary coproducts and kernels of split epis. The category of internal actions in $\mathbf{B}$, denoted $Act(\mathbf{B})$, is defined as follows.

Objects are triples (X, ξ, B) where X and B are objects in $\mathbf{B}$ and $\xi : B\flat X \longrightarrow X$ is a morphism such that

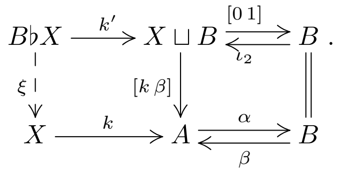
$$\begin{aligned} \xi\eta_X &= 1 \\ \xi\mu_X &= \xi(1\flat\xi) \end{aligned}$$

where the object $B\flat X$ is the kernel, $k : B\flat X \longrightarrow X \sqcup B$ of $[0, 1] : X \sqcup B \longrightarrow B$ and η_X, μ_x are induced, respectively, by ι_1 and $[k\ \iota_2]$. See [7] for more details.

We now have to consider $Act(\mathbf{B})$ as an half-reflection, (I, G) (with G admitting a left adjoint), over $\mathbf{B}$. Clearly we have a functor



sending a split epi (A, α, β, B) to (X, ξ, B) as suggested in the following diagram



It is well defined because k (being a kernel) is monic and

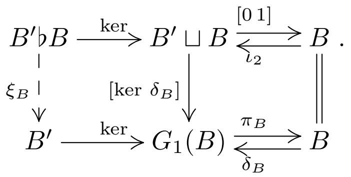
$$k\xi\eta_X = [k\ \beta]k'\eta_X = [k\ \beta]\iota_1 = k$$

so that $\xi\eta_X = 1$; a similar argument shows $\xi\mu_X = \xi(1\flat\xi)$.

To obtain a functor $G : \mathbf{B} \longrightarrow Act(\mathbf{B})$ we compose

$$\mathbf{B} \longrightarrow Pt(\mathbf{B}) \longrightarrow Act(\mathbf{B})$$

where $\mathbf{B} \longrightarrow Pt(\mathbf{B})$ is the half-reflection of Corollary 2; the resulting G sends an object $B \in \mathbf{B}$ to the internal action $(B'\flat B, \xi_B, B)$ as suggested by the following diagram



In the case of Groups this corresponds to the action by conjugation (see [7]). The next step is to require that G admits a left adjoint, which in the case of Groups is true and it corresponds to the construction of a semi-direct product from a given action.

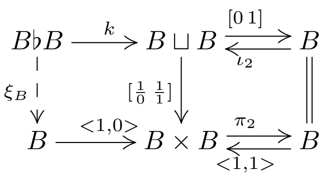
For convenience, we will now assume the existence of binary products, instead of the data $G_1, \pi, \delta, \varepsilon$ of Theorem 10.

For the rest of this section, and if not explicitly stated otherwise, we will assume that $\mathbf{B}$ is a pointed category with binary products and coproducts and kernels of split epis.

With those assumptions we may consider the half-reflection

$$Act(\mathbf{B}) \underset{G}{\overset{I}{\rightleftarrows}} \mathbf{B} \ ,\ \pi : 1 \longrightarrow GI$$

with $I(X, \xi, B) = B$ and $G(B) = (B, \xi_B, B)$ where ξ_B is obtained as a restriction to the kernels



(note that $\langle 0,1\rangle$ is the kernel of π_2), and the natural transformation $\pi : 1 \longrightarrow GI$ is given by

$$\begin{array}{ccc} (X,\xi,B) & & B\flat X \xrightarrow{\ \xi\ } X \\ \downarrow{\scriptstyle (0,1)} & & {\scriptstyle 1\flat 0}\downarrow \qquad\quad \downarrow{\scriptstyle 0} \\ (B,\xi_B,B) & & B\flat B \xrightarrow{\ \xi_B\ } B \end{array} ;$$

which is well defined because $\xi_B\,(1\flat 0) = 0$, since $\langle 1,0\rangle\,\xi_B\,(1\flat 0) = \langle 0,0\rangle$:

$$\langle 1,0\rangle\,\xi_B\,(1\flat 0) = [\langle 1,0\rangle\ \ \langle 1,1\rangle]\,(0\sqcup 1)\ker[0\ 1] =$$
$$= \langle [1\ 1],[0\ 1]\rangle\,(0\sqcup 1)\ker[0\ 1] = \langle 0,[0\ 1]\rangle\ker[0\ 1] = \langle 0,0\rangle\,.$$

Definition 6 (semi-direct products)**.** We will say that $\mathbf{B}$ has semidirect products, if the functor G admits a left adjoint.

Note that this is a weaker notion than the one introduced in [5], since we are not asking that the induced adjunction between $Act(\mathbf{B})$ and $Pt(\mathbf{B})$ to be an equivalence of categories.

We now state a sufficient condition for $\mathbf{B}$ to have semidirect products in the sense of the definition above.

Theorem 11. *The functor G, in the half-reflection*

$$Act\,(\mathbf{B}) \underset{G}{\overset{I}{\rightleftarrows}} \mathbf{B}\ \ ,\ \ \pi : 1 \longrightarrow GI,$$

as above, admits a left adjoint if the category $\mathbf{B}$ *has coequalizers of reflexive graphs.*

Proof. Given an object (X,ξ,B), consider the reflexive graph

$$(B\flat X)\sqcup B \underset{\xi\sqcup 1}{\overset{[k\ \iota_2]}{\rightrightarrows}} X\sqcup B\ \ \ (\text{with } \eta\sqcup 1 : X\sqcup B \to (B\flat X)\sqcup B)\,.$$

The left adjoint, F, is given by the coequalizer of $[k\ \iota_2]$ and $\xi\sqcup 1$:

$$(B\flat X)\sqcup B \underset{\xi\sqcup 1}{\overset{[k\ \iota_2]}{\rightrightarrows}} X\sqcup B \xrightarrow{\ \sigma\ } F(X,\xi,B)\ .$$

See [7] for more details. ☐

Let us from now on assume that $\mathbf{B}$ is a pointed category with binary products and coproducts and kernels of split epis, and coequalizers of reflexive graphs.

The next step is to consider the canonical functor

$$Act\,(\mathbf{B}) \xrightarrow{\ T\ } Pt\,(\mathbf{B})$$

sending (X,ξ,B) to the split epi $\left(F\,(X,\xi,B)\,,\overline{[0\ 1]},\sigma\iota_2\right)$ where $\overline{[0\ 1]}$ is such that $\overline{[0\ 1]}\sigma = [0\ 1]$, investigate whether it is an equivalence of categories and study internal precategories and reflexive graphs in $\mathbf{B}$.

First we show that under the given assumptions, it is always an adjunction.

Theorem 12. *The functors*

$$Act\,(\mathbf{B}) \underset{S}{\overset{T}{\rightleftarrows}} Pt\,(\mathbf{B})$$

as defined above, form an adjoint situation.

Proof. Consider the following diagram

$$\begin{array}{ccccc} B\flat X \xrightarrow{\ \xi\ } X & : & X \xrightarrow{\ \sigma\iota_1\ } F(X,B) & \underset{\sigma\iota_2}{\overset{\overline{[0\ 1]}}{\rightleftarrows}} & B \\ {\scriptstyle f_0\flat g}\downarrow \qquad\quad \downarrow{\scriptstyle g} & & {\scriptstyle g}\downarrow \qquad\qquad \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ B'\flat X' \xrightarrow{\ \xi'_A\ } X' & & X' \xrightarrow{\ \ k\ \ } A & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & B' \end{array} \tag{4.2}$$

where (X,ξ,B) is an object in $Act\,(\mathbf{B})$ and (X',ξ'_A,B') is $S\,(A,\alpha,\beta,B')$.

Given (f_1,f_0), since k is the kernel of α and

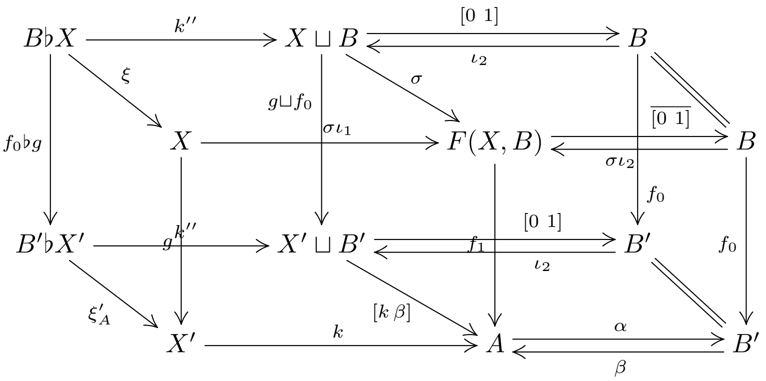
$$\alpha f_1\sigma\iota_1 = f_0\overline{[0\ 1]}\sigma\iota_1 = f_0[0\ 1]\iota_1 = 0$$

we obtain g as the unique morphism such that $kg = f_1\sigma\iota_1$.

To prove that the pair (g,f_0) is a morphism in $Act\,(\mathbf{B})$, that is, the left hand square in (4.2) commutes, we have to show

$$\xi'_A\,(f_0\flat g) = g\xi$$

and we do the following: first observe that $k\xi'_A\,(f_0\flat g) = f_1\sigma k''$, in fact (see diagram below, where k' and k'' are kernels)



$$\begin{aligned} k\xi'_A\,(f_0\flat g) &= [k\ \beta]k'\,(f_0\flat g)\ \text{, definition of } \xi'_A \\ &= [k\ \beta]\,(g\sqcup f_0)\,k'' \\ &= [kg\ \beta f_0]k'' \end{aligned}$$

and

$$\begin{aligned} f_1\sigma k'' &= f_1\sigma[\iota_1\ \iota_2]k'' \\ &= [f_1\sigma\iota_1\ f_1\sigma\iota_2]k'' \\ &= [kg\ \beta f_0]k''; \end{aligned}$$

we also have $kg\xi = f_1\sigma\iota_1\xi$, by definition of g. The result follows from the fact that k is monic and

$$\sigma k'' = \sigma\iota_1\xi,$$

which follows from (4.3) by taking $f = \sigma\iota_1$ and $g = \sigma\iota_2$.

Conversely, given g and f_0 such that the left hand square in (4.2) commutes, we find $f_1 = \overline{[kg\ \beta f_0]}$, which is well defined because (see 4.3 below)

$$kg\xi = [kg\ \beta f_0]k'',$$

indeed we have

$$\begin{aligned} kg\xi &= k\xi'_A\,(f_0\flat g) \\ &= [k\ \beta]k'\,(f_0\flat g) \\ &= [k\ \beta]\,(g\sqcup f_0)\,k'' \\ &= [kg\ \beta f_0]k''. \end{aligned}$$

□

In the following results we will need to use the observation that in order to give a morphism

$$F\,(X,\xi,B) \longrightarrow B'$$

it is the same as to give a pair (f,g) with $f : X \longrightarrow B'$ and $g : B \longrightarrow B'$ satisfying

$$[f\ g][k\ \iota_2] = [f\ g]\,(\xi\sqcup 1)$$

or, equivalently,

$$[f\ g]k = f\xi. \tag{4.3}$$

See [7] for more details.

Theorem 13. *Let $\mathbf{B}$ be a pointed category with binary products and coproducts, kernels of split epis and coequalizers of reflexive graphs. If the canonical functor*

$$Act\,(\mathbf{B}) \xrightarrow{T} Pt\,(\mathbf{B})$$

is an equivalence, then:

$$RG\,(\mathbf{B}) \sim \textit{Pre-X-Mod}\,(\mathbf{B})$$

$$PC\,(\mathbf{B}) \sim \textit{2-ChainComp}\,(\mathbf{B})\,.$$

The objects in Pre-X-Mod$(\mathbf{B})$ are pairs (h,ξ) with $h : X \longrightarrow B$ a morphism in $\mathbf{B}$ and $\xi : B\flat X \longrightarrow X$ an action (X,ξ,B) in $Act\,(\mathbf{B})$ satisfying the following condition

$$[h\ 1]k = h\xi,$$

with $k : B\flat X \longrightarrow X\sqcup B$ the kernel of $[0\ 1]$;
The objects in 2-ChainComp$(\mathbf{B})$ are sequences

$$Z \xrightarrow{t} X \xrightarrow{h} B \quad , \quad ht = 0$$

together with actions

$$\begin{aligned} \xi_X &: B\flat X \longrightarrow X \\ \xi_Z &: X\flat Z \longrightarrow Z \\ \xi_{F(Z,X)} &: F(X,B)\flat F(Z,X) \longrightarrow F(Z,X) \end{aligned}$$

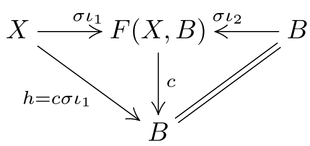
subject to the following conditions

$$\begin{aligned} [h\ 1]k_X &= h\xi_X \\ [t\ 1]k_Z &= t\xi_Z \\ [\sigma\iota_1\overline{[t\ 1]}\ 1]k_{F(Z,X)} &= \sigma\iota_1\overline{[t\ 1]}\xi_{F(Z,X)} \\ \overline{[0\ 1]}\xi_{F(Z,X)} &= \xi_X\left(\overline{[h\ 1]}\flat\overline{[0\ 1]}\right) \\ \sigma\iota_2\xi_X &= \xi_{F(Z,X)}\,(\sigma\iota_2\flat\sigma\iota_2)\,. \end{aligned}$$

Proof. Using the equivalence T, a reflexive graph in $\mathbf{B}$ is of the form

$$F(X,B) \underset{c}{\overset{\overline{[0\,1]}}{\rightrightarrows}} B \ (\text{with } \sigma\iota_2 : B \to F(X,B)) \qquad c\sigma\iota_2 = 1.$$

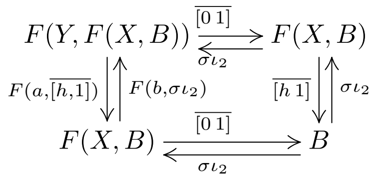
By definition of $F\,(X,B)$ we have



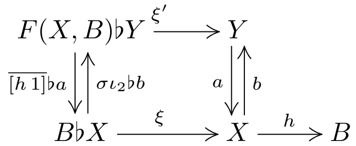
and the pair $(h,1)$ induces $c = \overline{[h\ 1]}$ if and only if

$$[h\ 1]k = h\xi.$$

For a precategory, observing that a split square (2.2) is in fact a split epi in $Pt\,(\mathbf{B})$ and using the equivalence $Act\,(\mathbf{B}) \sim Pt\,(\mathbf{B})$ we have that every such split square is of the form



and hence giving such a split square is to give internal actions (X,ξ,B) and $(Y,\xi',F\,(X,B))$ together with morphisms a,b,h such that the following squares commute



and

$$[h\ 1]k = h\xi.$$

It remains to insert the morphism

$$m : F\,(Y, F\,(X,B)) \longrightarrow F\,(X,B)$$

satisfying the following conditions

$$m\sigma\iota_2 = 1 \tag{4.4}$$

$$mF\,(b,\sigma\iota_2) = 1 \tag{4.5}$$

$$\overline{[h\ 1]}m = \overline{[h\ 1]}F\left(a,\overline{[h\ 1]}\right) \tag{4.6}$$

$$\overline{[0\ 1]}m = \overline{[0\ 1]}\overline{[0\ 1]}. \tag{4.7}$$

From (4.4) we conclude that $m = \overline{[v\ 1]}$ for some $v : Y \longrightarrow F(X, B)$ such that

$$[v\ 1]k' = v\xi'.$$

Using (4.7) we conclude that $\overline{[0\ 1]}v = 0$ so that v factors through the kernel of $\overline{[0\ 1]}$, which is $\sigma\iota_1$ because T is an equivalence, and finally we have

$$m = \overline{[\sigma\iota_1 u\ 1]}$$

for some $u : Y \longrightarrow X$ such that

$$[\sigma\iota_1 u\ 1]k' = \sigma\iota_1 u\xi'.$$

Condition (4.5) gives $ub = 1$ while condition (4.6) gives $ha = hu$:

$$\begin{aligned}
mF(b, \sigma\iota_2) = 1 &\Leftrightarrow \\
\overline{[\sigma\iota_1 u\ 1]}F(b, \sigma\iota_2) = 1 &\Leftrightarrow \\
\overline{[\sigma\iota_1 u\ 1]}\overline{[\sigma\iota_1 b\ \sigma\iota_2\sigma\iota_2]} = 1 &\Leftrightarrow \\
\overline{[\sigma\iota_1 u\ 1]}[\sigma\iota_1 b\ \sigma\iota_2\sigma\iota_2] = \sigma &\Leftrightarrow \\
[\sigma\iota_1 ub\ \ \sigma\iota_2] = [\sigma\iota_1\ \sigma\iota_2] &\Leftrightarrow \\
\sigma\iota_1 ub = \sigma\iota_1 &\Leftrightarrow \\
ub = 1&;
\end{aligned}$$

$$\begin{aligned}
\overline{[h\ 1]}m = \overline{[h\ 1]}F\left(a, \overline{[h\ 1]}\right) &\Leftrightarrow \\
\overline{[h\ 1]}\overline{[\sigma\iota_1 u\ 1]} = \overline{[h\ 1]}\overline{[\sigma\iota_1 a\ \ \sigma\iota_2\overline{[h\ 1]}]} &\Leftrightarrow \\
\overline{[h\ 1]}[\sigma\iota_1 u\ 1] = \overline{[h\ 1]}[\sigma\iota_1 a\ \ \sigma\iota_2\overline{[h\ 1]}] &\Leftrightarrow \\
\Leftrightarrow [hu\ \overline{[h\ 1]}] = [ha\ \overline{[h\ 1]}] &\Leftrightarrow \\
hu = ha&.
\end{aligned}$$

Conclusion 1: A precategory in $\mathbf{B}$ is given by the following data

$$\begin{array}{ccccc}
F(X,B)\flat Y & \xrightarrow{\xi'} & Y & & \\
{\scriptstyle \overline{[h\ 1]}\flat a}\downarrow\uparrow{\scriptstyle \sigma\iota_2\flat b} & & {\scriptstyle a}\downarrow\uparrow{\scriptstyle b}\ \searrow{\scriptstyle u} & & \\
B\flat X & \xrightarrow{\xi} & X & \xrightarrow{h} & B
\end{array} \tag{4.8}$$

such that ξ,ξ' are internal actions, the obvious squares commute, and the following conditions are satisfied

$$\begin{aligned}
hu &= ha \\
ub &= 1 = ab \\
[\sigma\iota_1 u\ 1]k' &= \sigma\iota_1 u\xi' \\
[h\ 1]k &= h\xi.
\end{aligned} \tag{4.9}$$

We now continue to investigate it further and replace the split epi (Y, a, b, X) with an action (Z, ξ_Z, B). For convenience we will also rename $\xi_X := \xi$, $k_X := k$, $\xi_{F(Z,X)} := \xi'$, $k_{F(Z,X)} = k'$. The diagram (4.8) becomes

$$\begin{array}{ccccc}
F(X,B)\flat F(Z,X) & \xrightarrow{\xi_{F(Z,X)}} & F(Z,X) & & \\
{\scriptstyle \overline{[h\ 1]}\flat\overline{[0\ 1]}}\downarrow\uparrow{\scriptstyle \sigma\iota_2\flat\sigma\iota_2} & & {\scriptstyle \overline{[0\ 1]}}\downarrow\uparrow{\scriptstyle \sigma\iota_2}\ \searrow{\scriptstyle \overline{[t\ 1]}} & & \\
B\flat X & \xrightarrow{\xi_X} & X & \xrightarrow{h} & B
\end{array}$$

for some $t : Z \longrightarrow X$ such that $[t\ 1]k_Z = t\xi_Z$. The commutativity of the appropriate squares in the diagram above plus the reinterpretation of conditions (4.9) gives the stated result. □

Remark 2. In order to consider a reflexive graph $(h : X \longrightarrow B, \xi : B\flat X \longrightarrow X)$ as a precategory of the form

$$\begin{array}{ccccc}
F(X,B)\flat X & \xrightarrow{\xi(\overline{[h\ 1]}\flat 1)} & X & & \\
{\scriptstyle \overline{[h\ 1]}\flat 1}\downarrow\uparrow{\scriptstyle \sigma\iota_2\flat 1} & & {\scriptstyle 1}\downarrow\uparrow{\scriptstyle 1}\ \searrow{\scriptstyle 1} & & \\
B\flat X & \xrightarrow{\xi} & X & \xrightarrow{h} & B
\end{array}$$

we need, in addition to $[h\ 1]k = h\xi$ that

$$[\sigma\iota_1\ 1]k' = \sigma\iota_1\xi\left(\overline{[h\ 1]}\flat 1\right)$$

where $k : B\flat X \longrightarrow X \sqcup B$ and $k' : F(X,B)\flat X \longrightarrow X \sqcup F(X,B)$ are the kernels of $[0\ 1]$.
In the case of Groups, this corresponds to the Peiffer identity that distinguishes a precrossed module from a crossed module.

We now give a characterization of categories $\mathbf{B}$ such that $Act(\mathbf{B}) \sim Pt(\mathbf{B})$.

Theorem 14. *Let $\mathbf{B}$ be a pointed category with binary products and coproducts, kernels of split epis and coequalizers of reflexive graphs. The canonical functor*

$$Act(\mathbf{B}) \xrightarrow{T} Pt(\mathbf{B})$$

is an equivalence if and only if the following two properties hold in $\mathbf{B}$:
(A1) for every diagram of the form

$$X \xrightarrow{\sigma\iota_1} F(X,B) \underset{\sigma\iota_2}{\overset{\overline{[0\ 1]}}{\rightleftarrows}} B \tag{4.10}$$

the morphism $\sigma\iota_1$ is the kernel of $\overline{[0\ 1]}$;
(A2) the split short five lemma holds.

Proof. Similar to Theorem 9 , see also [5]. □

5 The general case

Let

$$\mathbf{A} \underset{G}{\overset{I}{\rightleftarrows}} \mathbf{B}\ ,\ \pi : 1 \longrightarrow GI$$

be a half-reflection.

Define a new category, denoted by $\mathbf{A}_1$ as follows: Objects are pairs (A, u) with $A \in \mathbf{A}$ and

$$u : A \longrightarrow GIA$$

such that $I(u) = 1$.
A morphism $f : (A, u) \longrightarrow (A', u')$ is a morphism $f : A \longrightarrow A'$ in $\mathbf{A}$ such that

$$\begin{array}{ccc} A & \xrightarrow{u} & GIA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle GIf} \\ A' & \xrightarrow{u'} & GIA' \end{array} .$$

Define another category, denoted $\mathbf{A}_2$, as follows:
Objects are systems

$$((E, v), a, b, (A, u))$$

where (E, v) and (A, u) are objects in $\mathbf{A}_1$,

$$a : (E, v) \longrightarrow (A, u)$$

is a morphism in $\mathbf{A}_1$, and

$$b : A \longrightarrow E$$

is a morphism in $\mathbf{A}$ such that

$$ab = 1_A.$$

Let $\mathbf{A} \xrightarrow{T} \mathrm{Pt}(\mathbf{B})$ be any subcategory of $\mathrm{Pt}(\mathbf{B})$, not necessarily full. We consider subcategories of reflexive graphs and internal precategories, in $\mathbf{B}$, restricted to split epis from $T(\mathbf{A})$, and denote them respectively by $RG_{\mathbf{A}}(\mathbf{B})$ and $PC_{\mathbf{A}}(\mathbf{B})$.

In particular if the functor G, as above, admits a left adjoint $(F, G, \eta, \varepsilon)$ and F is faithful and injective on objects, then the canonical functor

$$\mathbf{A} \xrightarrow{T} Pt(\mathbf{B})$$

determines a subcategory of split epis. This means that we have:

(1) Reflexive graphs internal to $\mathbf{B}$, and restricted to the split epis in $T(\mathbf{A})$, denoted $RG_{\mathbf{A}}(\mathbf{B})$, are of the form

$$FA \underset{c}{\overset{\varepsilon_{IA}F(\pi_A)}{\rightrightarrows}} IA \quad \text{(with } I(\eta_A) : IA \longrightarrow FA\text{)}$$

with $cI(\eta_A) = 1_{IA}$, for some $A \in \mathbf{A}$.

(2) Internal precategories in $\mathbf{B}$, relative to split epis in $T(\mathbf{A})$, denoted by $PC_{\mathbf{A}}(\mathbf{B})$, are of the form displayed next (note that we are using π'_A as an abbreviation for $\varepsilon_{IA}F(\pi_A)$ and the same holds for π'_E)

$$F(E) \;\; \begin{array}{c} \xrightarrow{\pi'_E} \\ \xleftarrow{I(\eta_E)} \\ \xrightarrow{m} \\ \xleftarrow{F(b)} \\ \xrightarrow{F(a)} \end{array} \;\; IE = FA \;\; \begin{array}{c} \xrightarrow{\pi'_A} \\ \xleftarrow{I(\eta_A)} \\ \xrightarrow{c} \end{array} \;\; IA \tag{5.1}$$

for some

$$E \underset{b}{\overset{a}{\rightleftarrows}} A \quad , \quad ab = 1_A$$

in $\mathbf{A}$, satisfying the following conditions

$$\begin{aligned} cI(\eta_A) &= 1_{IA} && (5.2) \\ I(a) &= c && (5.3) \\ I(b) &= I(\eta_A) && (5.4) \\ mI(\eta_E) &= 1_{IE} && (5.5) \\ mF(b) &= 1_{IE} && (5.6) \\ cm &= cF(a) && (5.7) \\ \pi'_A m &= \pi'_A \pi'_E. && (5.8) \end{aligned}$$

We observe that c is determined by a, and (5.2) follows from (5.3), (5.4) and the fact that $ab = 1_A$. We will be interested in the notion of multiplicative graph, which is obtained by removing (5.7) and (5.8). In some cases we will be interested in removing (5.6) so that the definition can be transported, from $\mathbf{B}$ to $\mathbf{A}$, and it does not depend on whether or not G admits a left adjoint.

Theorem 15. *For a half-reflection*

$$\mathbf{A} \underset{G}{\overset{I}{\rightleftarrows}} \mathbf{B} \;\; , \;\; \pi : 1 \longrightarrow GI,$$

if the functor G admits a left adjoint

$$(F, G, \eta, \varepsilon),$$

and F is faithful and injective on objects, then

$$\begin{aligned} \mathbf{A}_1 &\cong RG_{\mathbf{A}}(\mathbf{B}) && (5.9) \\ \mathbf{A}_2^* &\cong PC_{\mathbf{A}}(\mathbf{B}) && (5.10) \end{aligned}$$

where $\mathbf{A}_2^*$ is the subcategory of $\mathbf{A}_2$ given by the objects

$$((E, v), a, b, (A, u))$$

such that

$$\begin{aligned} IE &= FA \\ I(a) &= \varepsilon_{IA}F(u) \\ vb &= \eta_A \\ G(\varepsilon_{IA}F(\pi_A))\pi_E &= G(\varepsilon_{IA}F(\pi_A))v. \end{aligned}$$

Proof. The isomorphism (5.9) is established by the adjuntion $(F, G, \eta, \varepsilon)$. Given

$$A \xrightarrow{u} GIA \;\; , \;\; I(u) = 1_{IA}$$

we obtain

$$FA \;\; \begin{array}{c} \xrightarrow{\pi'_A} \\ \xleftarrow{I(\eta_A)} \\ \xrightarrow{u'} \end{array} \;\; IA \tag{5.11}$$

where $\pi'_A = \varepsilon_{IA}F(\pi_A)$, $u' = \varepsilon_{IA}F(u)$ and

$$u'I(\eta_A) = 1_{IA} \Leftrightarrow I(u) = 1.$$

Conversely, given (5.11), we obtain A, since F is injective on objects, and

$$u = G(u')\eta_A.$$

The isomorphism (5.10) is obtained as follows:
Given (5.1), since F in injective on objects and faithful, we obtain

$$E \underset{b}{\overset{a}{\rightleftarrows}} A \quad , \quad ab = 1_A \quad , \quad IE = FA.$$

Now define on the one hand

$$u = G(c)\,\eta_A \ \ , \ \ v = G(m)\,\eta_E;$$

while on the other hand

$$c = \varepsilon_{IA} F(u) \ \ , \ \ m = \varepsilon_{IE} F(v)\,,$$

and we have the following translation of equations:

Eq. n.^{o}	in $\mathbf{B}$	in $\mathbf{A}$
5.2	$cI(\eta_A) = 1_{IA}$	$I(u) = 1_{IA}$
5.5	$mI(\eta_E) = 1_{IE}$	$I(v) = 1_{IE}$
5.7	$cm = cF(a)$	$ua = GI(a)\,v$
5.3	$I(a) = c$	$I(a) = \varepsilon_{IA} F(u)$
5.4 5.6	$I(b) = I(\eta_A)$ $mF(b) = 1_{IE}$	$vb = \eta_A$
5.8	$\pi'_A m = \pi'_A \pi'_E$	$G(\pi'_A)\,\pi_E = G(\pi'_A)\,v$

Note that $ua = GI(a)\,v$ follows from the fact that a is a morphism in $\mathbf{A}_1$, to the contrary of b, which is just a morphism in $\mathbf{A}$. □

5.1 One more ingredient

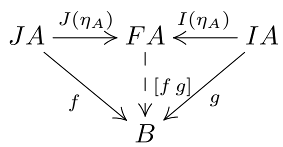
In some cases we also have a functor

$$J : \mathbf{A} \longrightarrow \mathbf{B}$$

satisfying the following three conditions:

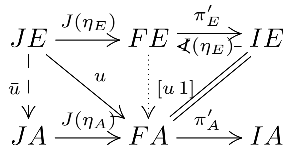
(1) $JG = 1_{\mathbf{B}}$

(2) the pair $(J(\eta_A), I(\eta_A))$ is jointly epic for every $A \in \mathbf{A}$, that is, given a pair of morphisms (f, g) as displayed below



there is at most one morphism $\alpha : FA \longrightarrow B$, with the property that $\alpha J(\eta_A) = f$ and $\alpha I(\eta_A) = g$, denoted by $\alpha = [f, g]$ when it exists. The pair (f, g) is said to be admissible (or cooperative in the sense of [4]) w.r.t. A, if $[f, g]$ exists.

(3) for every $A, E \in \mathbf{A}$, with $IE = FA$, a morphism, $u : J(E) \longrightarrow FA$, such that $(u, 1_{IE})$ is cooperative w.r.t. E and satisfying $\pi_A[u\ 1] = \pi_A \pi_E$, always factors trough $J(A)$, i.e., given u as in the diagram below



such that $[u\ 1]$ exists and $\pi_A[u\ 1] = \pi_A \pi_E$ then $u = J(\eta_A)\,u'$ for a unique $u' : JE \longrightarrow JA$.

Theorem 16. *Let* $\mathbf{B}$ *be a category, with* (I, G, π)

$$\mathbf{A} \underset{G}{\overset{I}{\rightleftarrows}} \mathbf{B} \ \ , \ \ \pi : 1 \longrightarrow GI,$$

a half-reflection such that the functor G *admits a left adjoint*

$$(F, G, \eta, \varepsilon)\,.$$

If we can find a functor

$$J : \mathbf{A} \longrightarrow \mathbf{B}$$

as above, then

- *the category* $RG_{\mathbf{A}}(\mathbf{B})$ *of reflexive graphs in* $\mathbf{B}$ *relative to split epis from* $\mathbf{A}$*, is given by:*
  Objects are pairs (A, h)*, with* $A \in \mathbf{A}$*, and* $h : JA \longrightarrow IA$ *a morphism such that* $(h, 1_{IA})$ *is admissible w.r.t.* A*;*
  A morphism $f : (A, h) \longrightarrow (A', h')$ *is a morphism* $f : A \longrightarrow A'$ *in* $\mathbf{A}$ *such that* $h'F(f) = I(f)\,h$.

- *the category of internal precategories in* $\mathbf{B}$ *relative to split epis from* $\mathbf{A}$*,* $PC_{\mathbf{A}}(\mathbf{B})$*, is given by:*
  Objects:
  $$(A, E, a, b, t, h)$$
  where $A, E,$ *are objects in* $\mathbf{A}$ *, with* $IE = FA$*,* $a, b, t, h,$ *are morphisms in* $\mathbf{B}$*,*
  $$JE \underset{t}{\overset{a}{\rightrightarrows}} JA \xrightarrow{\ h\ } IA \qquad (b : JA \to JE)$$
  such that
  $$ab = 1 = tb \ \ , \ \ ha = ht$$
  the pairs $(h, 1_{IA})$ *and* $(J(\eta_E)\,b, I(\eta_E)\,I(\eta_A))$ *are admissible w.r.t.* A*, and the pairs* $(J(\eta_A)\,a, I(\eta_A)\,[h\ 1])$ *and* $\left(J(\eta_A)\,t, 1_{I(E)}\right)$ *are admissible w.r.t.* E*.*
  Morphisms are triples (f_3, f_2, f_1) *of morphisms*
  $$\begin{array}{ccccc} JE & \underset{t}{\overset{a}{\rightrightarrows}} & JA & \xrightarrow{h} & IA \\ \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} \\ JE' & \underset{t'}{\overset{a'}{\rightrightarrows}} & JA' & \xrightarrow{h'} & IA' \end{array}$$
  such that the obvious squares in the above diagram commute and furthermore the pair

$(J(\eta_{A'}) f_2, I(\eta_{A'}) f_1)$ is admissible w.r.t. A while the pair $(J(\eta_{E'}) f_3, I(\eta_{E'}) [J(\eta_{A'}) f_2 \;\; I(\eta_{A'}) f_1])$ is admissible w.r.t. E.

Proof. Calculations are similar to the previous sections and the resulting diagram (5.1) is given by

$$\begin{aligned} c &= [h\ 1] \\ m &= [J(\eta_A)\, t \;\; 1_{I(E)}] \\ F(b) &= [J(\eta_E)\, b \;\; I(\eta_E) I(\eta_A)] \\ F(a) &= [J(\eta_A)\, a \;\; I(\eta_A)[h\ 1]]. \end{aligned}$$

The same argument applies to obtain the morphisms. □

5.2 The example of unitary magmas with right cancellation

An example of a general situation in the conditions of the above theorem is the following one.

Let $\mathbf{B}$ be a pointed category with kernels of split epis, with binary products and coproducts and such that the pair $(\langle 1,0\rangle, \langle 1,1\rangle)$, as displayed

$$B \xrightarrow{<1,0>} B \times B \underset{<1,1>}{\overset{\pi_2'}{\rightleftarrows}} B$$

is jointly epic for every $B \in \mathbf{B}$, and then consider: $\mathbf{A}$, the full subcategory of Pt($\mathbf{B}$) given by the split epis with the property that

$$X \xrightarrow{\ker \alpha} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B$$

the pair $(\ker \alpha, \beta)$ is jointly epic (identifying (A, α, β, B) with (A', α', β', B) whenever $A \cong A'$, in order to obtain F injective on objects).

Then we have functors

$$\begin{aligned} I, F, J &: \mathbf{A} \longrightarrow \mathbf{B} \\ G &: \mathbf{B} \longrightarrow \mathbf{A} \end{aligned}$$

with

$$\begin{aligned} I(A,\alpha,\beta,B) &= B \\ F(A,\alpha,\beta,B) &= A \\ J(A,\alpha,\beta,B) &= X \quad \text{, the object kernel of } \alpha \\ G(B) &= (B \times B, \pi_2, \langle 1,1\rangle, B) \end{aligned}$$

and with $\pi : 1_{\mathbf{A}} \longrightarrow GI$ given by $\pi = [0\ 1]$.

An example of such a category is the category of unitary magmas with right cancellation. Also every strongly unital category satisfies the above requirements (see [2], and references there).

The author thanks to Professors G. Janelidze and D. Bourn for much appreciated help of various kinds.

Acknowledgement This research work was supported by the Portuguese Foundation for Science and Technology (FCT) through the Project reference UID/ Multi/04044/2013, post doctoral grant SFRH/BPD/43216/2008 at CMUC, projects PTDC/MAT/120222/2010 and PTDC/EME-CRO/120585/2010, and also by CDRSP and ESTG from the Polytechnic Institute of Leiria.

References

[1] O. Berndt, "A categorical definition of semidirect products", App. Cat. Struct., Vol 6, 1998, 37–62.

[2] F. Borceux and D. Bourn, *Mal'cev, Protomodular, Homological and Semi-Abelian Categories*, Math. Appl. 566, Kluwer, 2004.

[3] D. Bourn, "Another denormalization theorem for the abelian chain complexes", Homology, Homotopy and Applications, Vol. 5(1), 2003, 49-52.

[4] D. Bourn and M. Gran, "Centrality and normality in protomodular categories", Theory and Applications of Categories, Vol. 9, 2002, 151–165

[5] D. Bourn and G. Janelidze, "Protomodularity, descent and semi-direct products", Theory and Applications of Categories, Vol. 4(2), 1998, pp.37-46

[6] R. Brown and Ph. J. Higgins, "Cubical Abelian Groups with Connections are Equivalent to Chain Complexes", Homology, Homotopy and Applications vol. 5(1), 2003, 49–52.

[7] Janelidze, G.: "Internal Crossed Modules", Georgian Math. J., 10, no. 1, 2003, 99–114.

[8] G. Janelidze, "Additive Categories", notes for the UCT Seminar on Category Theory, 2006.

[9] S. MacLane, *Categories for the Working Mathematician*, Springer-Verlag, 1998, 2nd edition.

[10] N. Martins-Ferreira, "Internal precategories relative to split epis", arXiv:0903.0333, 2009, 31 pages.

[11] N. Martins-Ferreira, "Crossed modules and Whitehead sequences", Journal of Homotopy and Related Structures **11** (4) ,2016, 893–921.

[12] A. Patchkoria, "Crossed semimodules and Schreier internal categories in the category of Monoids", Georgian Math. J., Vol. 5, No. 6, 1998, 575-581.