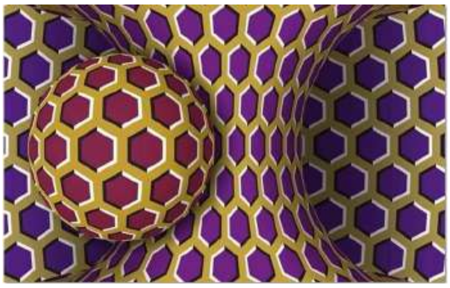
Scripta-Ingenia

Ordem, Monóides e Topologia



Alice Proverbio, optical illusion, Milan

Neste pequeno texto é assumido que o leitor conhece as bases da Teoria dos Conjuntos e está ciente de que a noção de *conjunto* não se define sendo antes um conceito primitivo da teoria, assim como a relação de pertença. Escrevemos por exemplo $x \in X$ para dizer que x é um elemento do conjunto X e usamos a notação $S \subseteq X$ para dizer que cada um dos elementos de S é também um elemento de X. Uma abordagem moderna da axiomática da Teoria dos Conjuntos pode ser encontrada em [Tom Leinster. (2014). Rethinking Set Theory. The American Mathematical Monthly, 121(5), 403-415. doi:10.4169/amer.math.monthly.121.05.403]. Dados dois conjuntos A e B obtemos o produto cartesiano $A \times B$ considerando que os seus elementos são todos os pares ordenados da forma (a, b) com $a \in A$, $b \in B$. Uma relação binária de A em B é um subconjunto do produto cartesiano. Se $R \subseteq A \times B$ é uma relação binária de A em B, escrevemos aRb em vez de $(a, b) \in R$.

Uma relação de (pré)-ordem num conjunto X é uma relação binária $R \subseteq X \times X$ tal que: (i) xRx para todo $x \in X$ (reflexividade); (ii) se xRy e yRz então xRz para todo $x, y, z \in X$ (transitividade).

A partir de um conjunto X dado, podemos formar o conjunto das partes de X, $\mathcal{P}(X)$, que tem como elementos cada um dos subconjuntos de X, incluindo o conjunto vazio, $\emptyset$, e o próprio X. Uma topologia em X é uma família de subconjuntos de X, $\tau \subseteq \mathcal{P}(X)$, cujos elementos se designam por *abertos*, e é tal que a interseção finita de *abertos* é um *aberto*, a união arbitrária de *abertos* é um *aberto* e ambos $\emptyset$ e X são *abertos*.

Um monoide é constituido por um sistema $(A, m, 0)$ no qual A é um conjunto, $0 \in A$ e $m \colon A \times A \to A$ é uma operaçao binária em A que é associativa, $m(m(x, y), z) = m(x, m(y, z))$, e unitária, $m(x, 0) = x = m(0, x)$, para todo $x, y, z \in A$.

À partida parece não haver qualquer semelhança entre os três conceitos introduzidos, nomeadamente um conjunto pre-ordenado (X, R), um espaço topológico (X, τ) e um monoide $(A, +, 0)$, e no entanto, observamos o seguinte resultado.

Sejam $(X, +, 0)$ e $(A, +, 0)$ dois monóides e seja $B \subseteq X$ um subconjunto de X equipado com duas funções $s \colon A \times B \to B$ e $t \colon B \to A$ tais que: (i) $b_1 + s(t(b_1), b_2) \in B$, para todo $b_1, b_2 \in B$; (ii) para todo $a \in A$, $s(a, b_1 + s(t(b_1), b_2)) = s(a, b_1) + s(a + t(b_1), b_2)$; (iii) $s(a_1 + a_2, b) = s(a_1, s(a_2, b)) = s(a_2, s(a_1, b))$, para todo $a_1, a_2 \in A$, $b \in B$.

Nestas condições obtemos uma topologia τ em X definida por

$$\mathcal{O} \in \tau \Leftrightarrow \forall x \in \mathcal{O}, \exists a \in A, N(x, a) \subseteq \mathcal{O},$$

com $N(x, a) = \{x + s(a, b) \in X \mid b \in B\} \cup \{x\}$, e observamos que $N(x, a) \in \tau$ para todo $x \in X$ e $a \in A$. Além do mais, no caso em que $t(b) = 0$ para todo $b \in B$, obtemos, para cada $a \in A$, uma relação de pre-ordem em X definida por $xR_a y \Leftrightarrow y \in N(x, a)$.

Se considerarmos por exemplo $X = (\mathbb{Q}, +, 0)$, $A = (\mathbb{N}_0, +, 0)$, $B = \mathbb{Z}$, $t(b) = 0$ e $s(a, b) = bp^a$, com p um número primo qualquer, obtemos a topologia dita p-ádica. Por outro lado se considerarmos $X = (\mathbb{R}, +, 0)$, $A = (\mathbb{N}, \cdot, 1)$, $B =]-1, 1[$, $s(a, b) = \frac{b}{a}$ e $t(b) \in \mathbb{N}$ tal que $\frac{1}{t(b)} \leq 1 - |b|$ então obtemos a topologia usual na reta real.

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Director and Chief Editor — Nelson Martins-Ferreira
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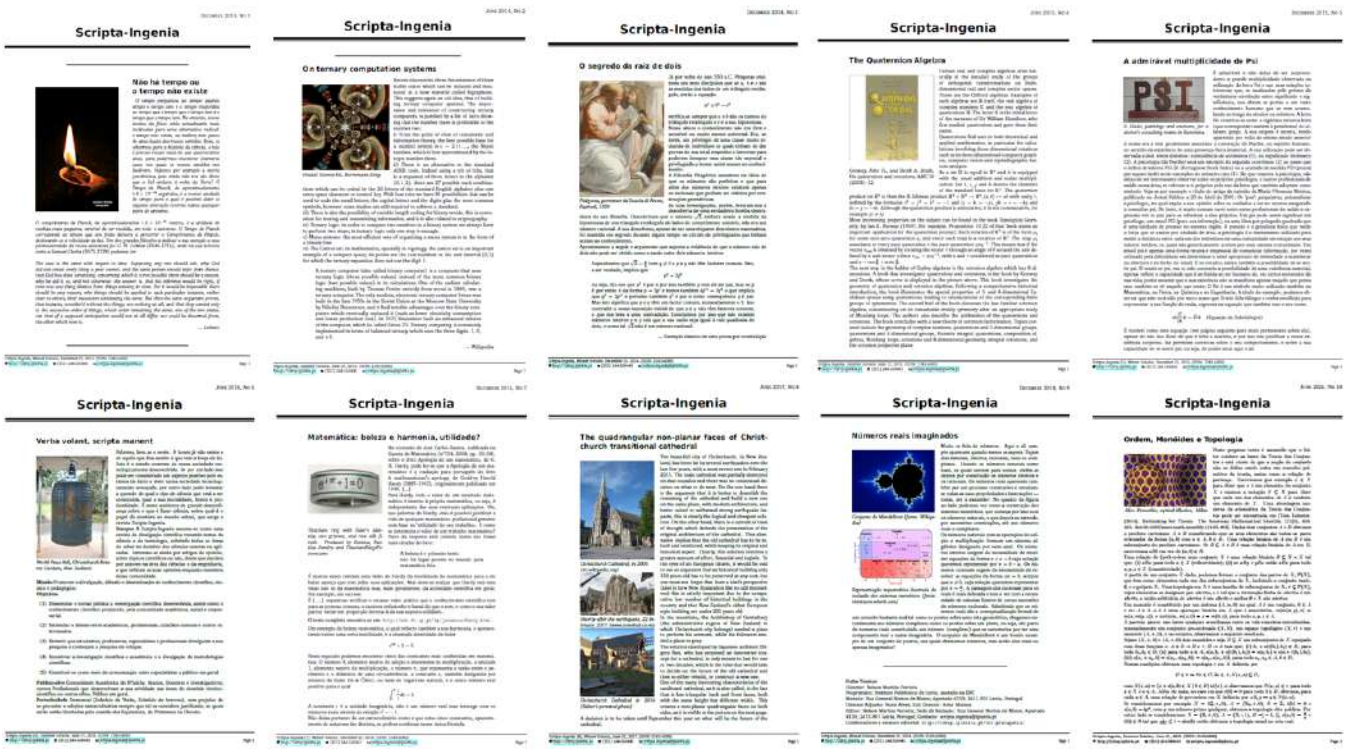

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A connection between unitary rings and mobi algebras

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Abstract A connection between the category of mobi algebras and the category of unitary rings is established. Although it is not a Galois connection, it shares some of its properties.

Introduction

An algebraic structure called mobi algebra [2] has been introduced as a model to axiomatize the unit interval of real numbers. It consists of a set together with a ternary operation and three constants. Every unitary ring with $\frac{1}{2}$ (the inverse of 2) gives rise to a mobi algebra via the ternary operation $p(a,b,c) = a - ba + bc$. However, not every mobi algebra structure is obtained in this way. For example the unit interval itself is not a ring. The purpose of this note is to establish a categorical correspondence between unitary rings and mobi algebras. This correspondence is not an adjunction and hence it cannot be a categorical Galois connection in the sense of [4]. Nevertheless, there are some common features that are similar to a Galois connection, namely that the two functors in the correspondence, say (F,G), satisfy $FGF = F$ and $GFG = G$. Moreover, the respective subcategories of fixed points are characterized on the one side as unitary rings with $\frac{1}{2}$ and on the other side as mobi algebras with 2.

Preliminaries

In this section we recall the well-known reflections from the category of semirings into the one of rings and further into the one of rings with ½

$$\mathbf{SRng} \xrightarrow{J} \mathbf{Rng} \xrightarrow{I} \mathbf{Rng}_{1/2} .$$

These well-known constructions provide useful notation for the similar ones that will be needed later on.

The ring with one-half $I(R)$ is defined for every ring R together with a surjective map

$$R \times \mathbb{N} \xrightarrow{q_R} I(R) , \quad (r,n) \mapsto q(r,n)$$

such that

$$q(r,n) = q(r',n') \Leftrightarrow \exists k \in \mathbb{N}, \quad r2^{n'+k} = r'2^{n+k}.$$

It is not difficult to see that q is uniquely determined (up to a unique isomorphism) as the quotient map for the congruence that defines it. The ring operations on $I(R)$ are defined as

$$q(r,n) + q(r',n') = q(r2^{n'} + r'2^{n}, n+n')$$
$$q(r,n) \cdot q(r',n') = q(rr', n+n').$$

We observe that the operations are well defined. Indeed, if $q(r_1,n_1) = q(r_2,n_2)$ and $q(r'_1,n'_1) = q(r'_2,n'_2)$ then

$$q(r_1 2^{n'_1} + r'_1 2^{n_1}, n_1 + n'_1) = q(r_2 2^{n'_2} + r'_2 2^{n_2}, n_2 + n'_2)$$

and

$$q(r_1 r'_1, n_1 + n'_1) = q(r_2 r'_2, n_2 + n'_2).$$

We also observe that if $f\colon R \to R'$ is a ring homomorphism and if $(r,n) \sim (r',n')$ then $(f(r),n) \sim (f(r'),n')$. This means that $I(f)$ is determined as the unique function that fits into the diagram

$$\begin{array}{ccc} R \times \mathbb{N} & \xrightarrow{q_R} & I(R) \\ \downarrow{\scriptstyle f\times 1_{\mathbb{N}}} & & \downarrow{\scriptstyle I(f)} \\ R' \times \mathbb{N} & \xrightarrow{q_{R'}} & I(R') \end{array} .$$

In a similar manner we define the functor

$$J\colon \mathbf{SRng} \to \mathbf{Rng}$$

by specifying, for every semi-ring S, the ring $J(S)$ together with a surjective map

$$S \times S \xrightarrow{q_S} J(S) ; \quad (u,v) \mapsto q(u,v)$$

such that

$$q(u,v) = q(u',v') \Leftrightarrow \exists k \in S, u + v' + k = v + u' + k$$

and considering the operations

$$q(u,v) + q(u',v') = q(u+u', v+v')$$
$$q(u,v) \cdot q(u',v') = q(uu' + vv', uv' + vu')$$

which make $J(S)$ a unitary ring. Again, for any homomorphism of semi-rings $f\colon S \to S'$, we define $J(f)$ as the unique map that fits into the diagram

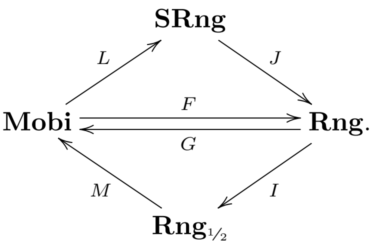
$$\begin{array}{ccc} S \times S & \xrightarrow{q_S} & J(S) \\ \downarrow{\scriptstyle f\times f} & & \downarrow{\scriptstyle J(f)} \\ S' \times S' & \xrightarrow{q_{S'}} & J(S') \end{array} .$$

A pre-Galois connection

In this section we define two functors

$$\mathbf{Mobi} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{Rng}$$

between mobi algebras and unitary rings by completing the following diagram with the functors M and L and then defining $F = JL$ and $G = MI$



The functor M is described in the paper [2] and it associates to every ring with ½ the mobi algebra structure

$$M(R) = (R, p, 0, 1/2, 1)$$

with

$$p(a, b, c) = (1 - b)a + bc.$$

If $f\colon R \to R'$ is a ring homomorphism then it is a mobi algebra homomorphism as well.

The functor L is constructed in a similar manner as the functor I. For every mobi algebra $A = (A, p, 0, 1/2, 1)$, the semi-ring $L(A)$ is defined together with a surjective map

$$A \times \mathbb{N} \xrightarrow{q_A} L(A)$$

such that

$$q(a, n) = q(a', n') \Leftrightarrow \exists k \in \mathbb{N}, \quad a \cdot (1/2)^{n'+k} = a' \cdot (1/2)^{n+k}$$

with the operation $\cdot$ defined as $a \cdot b = p(0, a, b)$. The semiring operations are given by the following formulas

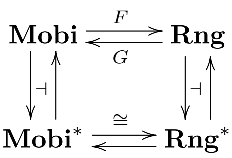
$$q(a, n) \cdot q(a', n') = q(a \cdot a', n + n')$$
$$q(a, n) + q(a', n') = q(a \cdot (1/2)^{n'} \oplus a' \cdot (1/2)^{n}, n + n' + 1),$$

with the operation $\oplus$ defined as $a \oplus b = p(a, 1/2, b)$.

Let us now see some simple concrete examples. Let $[0, 1]$ be the unit interval equipped with the canonical mobi algebra, that is $p(a, b, c) = a - ba + bc$ and the three constants $0, 1, 1/2$. Let $\mathbb{R}_0^+$ denote the semiring of non-negative real numbers and let $\mathbb{Z}$ and $\mathbb{R}$ be the usual rings of integers and real numbers. If denoting by $\mathbb{D}$ the ring of dyadic numbers (which is a ring with ½) then we have

$$L([0, 1]) = \mathbb{R}_0^+, \quad F([0, 1]) = \mathbb{R}, \quad I(\mathbb{Z}) = \mathbb{D}, \quad M(\mathbb{D}) = \mathbb{D}.$$

We now consider the diagram



in which $\mathbf{Mobi}^*$ is the full subcategory consisting on those mobi algebras A such that $A = G(B)$ for some ring B. In the same way, $\mathbf{Rng}^*$ is the full subcategory consisting of those unitary rings R such that $R = F(A)$ for some mobi algebra A. It turns out that $\mathbf{Mobi}^*$ is precisely $\mathbf{Mobi}_2$ while $\mathbf{Rng}^*$ is precisely $\mathbf{Rng}_{1/2}$. Moreover, it is straightforward to observe that $FGF = G$ and $GFG = F$. This observations should be compared with Theorems 3.8 and 4.2 in [4].

Conclusion

We conclude with the observation that the functor G cannot be a left adjoint to the functor F. Indeed, the inclusion $\mathbb{Z} \to \mathbb{R} = F([0, 1])$ does not have a counterpart as a morphism $G(\mathbb{Z}) = \mathbb{D} \to [0, 1]$. Moreover, the functor F cannot be a left adjoint to the functor G either. To see it let us first point out that if R is a ring with ½ then $I(R) \cong R$. This simply follows because in that case we find $q_R(r, n) = r(1/2)^n$. Furthermore, if $2 = 1 + 1 = 0$ (as in Boolean rings) then $I(R) = \{0\}$ is the trivial ring with $0 = 1$. Indeed, in this case we have

$$(r, n) \sim (r', n') \Leftrightarrow r2^{n'+k} = r'2^{n+k} \Leftrightarrow 0 = 0$$

and everything collapses in the same equivalence class. This means that there is no ring homomorphism between $\mathbb{R}$ and the two element (boolean) ring $\{0, 1\}$, for if $f\colon \mathbb{R} \to \{0, 1\}$ would be a ring homomorphism then we would have

$$f(1) = f(2 \cdot 1/2) = f(2) \cdot f(1/2) = 0 \cdot f(1/2) = 0 \neq 1.$$

This means that there is no counterpart $F([0, 1]) = \mathbb{R} \to \{0, 1\}$ to the unique homomorphism $[0, 1] \to \{0\} = G(\{0, 1\})$. And so the functor F is not a left adjoint to the functor G.

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Boolean algebras and mobi algebras

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Abstract A structure with a ternary operation that can give both Boolean and mobi algebras is presented. It is shown how each of these algebras originates from the ternary structure and why they are incompatible.

1 Introduction

In the mid-twentieth century, structures with a ternary operation have been introduced from which Boolean algebras are obtained ([4], [3], [7], see also [8], [2], [6]). They all can be seen as a set S with an operation $p : S\times S\times S \to S$ and two fixed elements 0 and 1 verifying some postulates. The existence and uniqueness, for each $a \in S$, of a complement $\overline{a}$ were predominantly guaranteed without the explicit use of the ternary operation p. Two of the postulates are those of median algebras (see [1] for a survey) defined, following [7], as a set closed under a ternary operation p satisfying:

$$p(a, a, b) = a \quad (1.1)$$
$$p(p(a, b, c), p(a, b, d), e) = p(p(c, d, e), a, b). \quad (1.2)$$

It is shown in [7] that a ternary operation satisfying these two postulates is completely commutative which means that

$$p(a, b, c) = p(b, a, c) = p(c, a, b). \quad (1.3)$$

This implies that in a median algebra we also have

$$p(a, b, a) = a \quad (1.4)$$
$$p(p(a, c, b), e, p(a, d, b)) = p(a, p(c, e, d), b). \quad (1.5)$$

Nevertheless the reverse is not true: a structure that verify (1.4) and (1.5) is more general than a median algebra and is not necessarily commutative. This difference is easily perceptible if we set $c = d$ in the above conditions. Indeed, using (1.1), (1.2) becomes $p(a, b, c) = p(c, a, b)$ while, using (1.4), (1.5) reads $p(a, c, b) = p(a, c, b)$.

Recently, it was introduced an algebraic structure, called mobi algebra [5], with a ternary operation verifying (1.4) and (1.5). Unary and binary operations can be derived and, within a mobi algebra, there exist involutions and monoids. Amongst the properties of the derived operations, relations similar to those of Boolean algebras arise naturally, as for instance the De Morgan laws. Nevertheless, mobi algebras have no imposed commutativity and therefore certainly differ from lattices. Moreover, the complement is deduced from the ternary operation instead of being independent of it.

The purpose of this article is to present a general ternary structure (see Section 2) that contains both mobi algebras (see Section 3) and Boolean algebras (see Section 4), to show how each of them can be obtained from it and to explain how they diverge to conflicting points. In particular, the binary operations that originate Boolean algebras under some conditions cannot be idempotent under the conditions of a mobi algebra. In Section 5, we point out some differences between this new way of generating Boolean algebras and the previous one that uses a commutative ternary operation. Appendix A gives all the possible algebras with 4 elements verifying Definition 1. This gives some hints on the kind of other structures encompassed besides mobi and boolean algebras. Appendix B presents the list of n-ary operations, with $n = 0, 1, 2$, constructed from a single ternary operation p verifying Definition 1.

2 General ternary structure

A ternary structure that contains mobi and Boolean algebras is presented. It verifies the postulates (1.4) and (1.5). An important aspect of (1.5) is that all three elements of the ternary operation are affected by the operation itself and consequently (1.5) can be interpreted as a condition for the existence of a certain automorphism. Let S be a set closed under a ternary operation p and let $a, b \in S$ be two fixed elements. On S, equipped with the binary operations p_c, defined for every $c \in S$ by $p_c(d, e) = p(d, c, e)$, the map $f : S \to S; c \mapsto p_c(a, b)$ is a homomorphism (considering that a and b are fixed) if and only if, for all $c, d, e \in S$

$$f(p_e(c, d)) = p_e(f(c), f(d))$$

which is exactly what (1.5) states.

Definition 1. *Let $(A, p, 0, 1)$ be a system in which A is a set, p is a ternary operation and 0 and 1 are elements of A, that satisfies the following axioms:*

(G1) $p(0, a, 1) = a$

(G2) $p(a, 0, b) = a$

(G3) $p(a, 1, b) = b$

(G4) $p(a, b, a) = a$

(G5) $p(a, p(c_1, c_2, c_3), b) = p(p(a, c_1, b), c_2, p(a, c_3, b))$.

Remark: Axiom **(G4)**, when together with the other four axioms, could be replaced by $p(0,b,0) = 0$ or $p(1,b,1) = 1$, because then

$$\begin{aligned} p(a,b,a) &= p(p(a,0,1),b,p(a,0,1)) \\ &= p(a,p(0,b,0),1) = p(a,0,1) = a \end{aligned}$$

or

$$\begin{aligned} p(a,b,a) &= p(p(0,1,a),b,p(0,1,a)) \\ &= p(0,p(1,b,1),a) = p(0,1,a) = a. \end{aligned}$$

Of particular interest in the algebras of Definition 1 are the unary operation "$\overline{()}$" and the binary operations "$\cdot$" and "$\circ$", defined as follows.

Definition 2. *Let $(A,p,0,1)$ be an algebra satisfying Definition 1. We define:*

$$\overline{a} = p(1,a,0) \quad (2.1)$$
$$a \cdot b = p(0,a,b) \quad (2.2)$$
$$a \circ b = p(a,b,1). \quad (2.3)$$

The canonical example is:
$A = [0,1]$, $p(a,b,c) = a - ba + bc$

$\overline{a} = 1 - a$

$a \cdot b = ab$

$a \circ b = a + b - ab.$

Let us begin with properties of the unary operation. From axioms **(G2)** and **(G3)**, we immediately find that:

$$\overline{1} = 0; \quad \overline{0} = 1. \quad (2.4)$$

Proposition 1. *Let $(A,p,1,0)$ be an algebra as in Definition 1. The following properties hold:*

$$\overline{\overline{a}} = a \quad (2.5)$$
$$p(b,c,a) = p(a,\overline{c},b) \quad (2.6)$$
$$\overline{p(a,c,b)} = p(\overline{a},c,\overline{b}). \quad (2.7)$$

Proof. To prove property (2.5), we use axioms **(G5)**, **(G3)**, **(G2)**, **(G1)**:

$$\begin{aligned} \overline{\overline{a}} &= p(1,p(1,a,0),0) \\ &= p(p(1,1,0),a,p(1,0,0)) = p(0,a,1) = a. \end{aligned}$$

To prove property (2.6), we use axioms **(G5)**, **(G3)**, **(G2)**

$$\begin{aligned} p(a,\overline{c},b) &= p(a,p(1,c,0),b) \\ &= p(p(a,1,b),c,p(a,0,b)) = p(b,c,a). \end{aligned}$$

Property (2.7) is a particular case of **(G5)**:

$$\begin{aligned} \overline{p(a,c,b)} &= p(1,p(a,c,b),0) \\ &= p(p(1,a,0),c,p(1,b,0)) \\ &= p(\overline{a},c,\overline{b}). \end{aligned}$$

□

Properties (2.6) and (2.7) can be combined to give:

$$\overline{p(a,b,c)} = p(\overline{c},\overline{b},\overline{a}) \quad (2.8)$$

From (2.4), (2.5) and (2.8), we can deduce the following De Morgan relations:

$$\overline{a \cdot b} = \overline{b} \circ \overline{a} \quad (2.9)$$
$$\overline{a \circ b} = \overline{b} \cdot \overline{a}. \quad (2.10)$$

The next propositions show how, within the structure of Definition 1, the two binary operations originate monoids.

Proposition 2. *$(A,\cdot,1)$ is a monoid:*

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad (2.11)$$
$$a \cdot 1 = a = 1 \cdot a. \quad (2.12)$$

Proof. Associativity is proven using **(G5)** and **(G2)**:

$$\begin{aligned} (a \cdot b) \cdot c &= p(0,p(0,a,b),c) \\ &= p(p(0,0,c),a,p(0,b,c)) = a \cdot (b \cdot c). \end{aligned}$$

Right identity is just **(G1)** and left identity is **(G3)**. □

Proposition 3. *$(A,\circ,0)$ is a monoid:*

$$a \circ (b \circ c) = (a \circ b) \circ c \quad (2.13)$$
$$a \circ 0 = a = 0 \circ a. \quad (2.14)$$

Proof. Associativity is proven using **(G5)** and **(G3)**:

$$\begin{aligned} a \circ (b \circ c) &= p(a,p(b,c,1),1) \\ &= p(p(a,b,1),c,1) = (a \circ b) \circ c. \end{aligned}$$

Left identity is just **(G1)** and right identity is **(G2)**. □

Using **(G4)**,**(G2)** and **(G3)**, we also get directly:

$$a \cdot 0 = 0 = 0 \cdot a \qquad a \circ 1 = 1 = 1 \circ a. \quad (2.15)$$

We end this section with finite examples of algebras as presented in Definition 1. There are only two such algebras with 3 elements. Let us call $0, X$ and 1 the three distinct elements. Properties (2.4) and (2.5) dictate that $\overline{X} = X$, and consequently (2.6) imply $p(a,X,b) = p(b,X,a)$. Therefore, Axioms **(G1)** to **(G3)** leave two undetermined constants, represented by α and β, in the ternary operation:

p (-,0,-)	0	X	1
0	0	0	0
X	X	X	X
1	1	1	1

p (-,X,-)	0	X	1
0	0	α	X
X	α	X	β
1	X	β	1

p (-,1,-)	0	X	1
0	0	X	1
X	0	X	1
1	0	X	1

which means that:

p (0,-,-)	0	X	1
0	0	0	0
X	0	α	X
1	0	X	1

p (-,-,1)	0	X	1
0	0	X	1
X	X	β	1
1	1	1	1

-	p (1,-,0)
0	1
X	X
1	0

.

Property (2.7) implies $\overline{\alpha} = \overline{p(0,X,X)} = p(1,X,X) = \beta$. Finally, we have for instance $p(0,p(1,X,X),X) = p(p(0,1,X),X,p(0,X,X))$, by Axiom **(G5)**, implying $p(0,\overline{\alpha},X) = p(X,X,\alpha)$, which excludes the case $\alpha = 0$. We are then left with 2 possibilities:

1. $\alpha = 1$ and $\beta = 0$ which is the smallest non-trivial mobi algebra [5].
2. $\alpha = X = \beta$ which is the only lattice with 3 elements (a chain).

3 Mobi algebras

Definition 3. *A mobi algebra [5] is a system $(A,p,1,\frac{1}{2},0)$, in which A is a set, p is a ternary operation and 1, $\frac{1}{2}$ and 0 are elements of A, that satisfy the following axioms:*

(A1) $p(1,\frac{1}{2},0) = \frac{1}{2}$

(A2) $p(0,a,1) = a$

(A3) $p(a,b,a) = a$

(A4) $p(a,0,b) = a$

(A5) $p(a,1,b) = b$

(A6) $p(a,\frac{1}{2},b) = p(a',\frac{1}{2},b) \implies a = a'$

(A7) $p(a,p(c_1,c_2,c_3),b) = p(p(a,c_1,b),c_2,p(a,c_3,b))$

(A8) $p(p(a_1,c,b_1),\frac{1}{2},p(a_2,c,b_2)) = p(p(a_1,\frac{1}{2},a_2),c,p(b_1,\frac{1}{2},b_2))$

Axioms **(A2)**,**(A3)**, **(A4)**, **(A5)** and **(A7)** show that a mobi algebra verifies Definition 1.

The existence of the element ½ is an important characteristic of mobi algebras. It is the unique element equal to its complement in the sense of Definition 2.

Proposition 4. *Let $(A,p,1,\frac{1}{2},0)$ be a mobi algebra. Then, the following properties hold:*

$$\overline{\tfrac{1}{2}} = \tfrac{1}{2} \quad (3.1)$$
$$p(a,\tfrac{1}{2},b) = p(b,\tfrac{1}{2},a) \quad (3.2)$$
$$p(\overline{a},\tfrac{1}{2},a) = \tfrac{1}{2} \quad (3.3)$$
$$\overline{a} = a \Rightarrow a = \tfrac{1}{2} \quad (3.4)$$
$$\tfrac{1}{2}\cdot\tfrac{1}{2} = \tfrac{1}{2} \Rightarrow a = b, \forall a,b \in A. \quad (3.5)$$

Proof. $\overline{\frac{1}{2}} = p(1,\frac{1}{2},0) = \frac{1}{2}$ by **(A1)**. (3.2) is a particular case of (2.6) considering that $\overline{\frac{1}{2}} = \frac{1}{2}$. The other properties are easily proved:

$$\begin{aligned} p(\overline{a},\tfrac{1}{2},a) &\overset{(A2)}{=} p(p(1,a,0),\tfrac{1}{2},p(0,a,1)) \\ &\overset{(A8)}{=} p(p(1,\tfrac{1}{2},0),a,p(0,\tfrac{1}{2},1)) \\ &\overset{(A1)}{=} p(\tfrac{1}{2},a,p(0,\tfrac{1}{2},1)) \\ &\overset{(A2)}{=} p(\tfrac{1}{2},a,\tfrac{1}{2}) \\ &\overset{(A3)}{=} \tfrac{1}{2} \end{aligned}$$

$$\begin{aligned} \overline{a} = a &\Rightarrow p(a,\tfrac{1}{2},a) = p(\overline{a},\tfrac{1}{2},a) \\ &\overset{(A3)}{\Rightarrow} a = p(\overline{a},\tfrac{1}{2},a) \\ &\overset{(3.3)}{\Rightarrow} a = \tfrac{1}{2} \end{aligned}$$

$$\begin{aligned} p(0,\tfrac{1}{2},\tfrac{1}{2}) &= \tfrac{1}{2} \\ &\Rightarrow p(a,p(0,\tfrac{1}{2},\tfrac{1}{2}),b) = p(a,\tfrac{1}{2},b) \\ &\overset{(A7)}{\Rightarrow} p(p(a,0,b),\tfrac{1}{2},p(a,\tfrac{1}{2},b)) = p(a,\tfrac{1}{2},b) \\ &\overset{(A4)}{\Rightarrow} p(a,\tfrac{1}{2},p(a,\tfrac{1}{2},b)) = p(a,\tfrac{1}{2},b) \\ &\overset{(A6)}{\Rightarrow} p(a,\tfrac{1}{2},b) = b \\ &\overset{(A3)}{\Rightarrow} p(a,\tfrac{1}{2},b) = p(b,\tfrac{1}{2},b) \\ &\overset{(3.2)}{\Rightarrow} p(b,\tfrac{1}{2},a) = p(b,\tfrac{1}{2},b) \\ &\overset{(A6)}{\Rightarrow} a = b \end{aligned}$$

□

In particular, (3.5) implies that, when 0, ½ and 1 are distinct elements, the binary operations of Definition 2 cannot be idempotent. There is a clear separation, within the structures of Definition 1, between mobi algebras and lattices.

4 Boolean algebras

The next proposition shows how to obtain a structure verifying the axioms of Definition 1 from a Boolean algebra and displays other properties verified by the corresponding ternary operation. The following propositions give sufficient conditions for the structure of Definition 1 to be a Boolean algebra.

Proposition 5. *Let $(A,\wedge,\vee,\overline{()},0,1)$ be a Boolean algebra and consider the ternary operation $p : A \times A \times A \to A$ defined by*

$$p(a,b,c) = (\overline{b} \wedge a) \vee (b \wedge c). \quad (4.1)$$

Then,

(i) $(A,p,0,1)$ is a system as in Definition 1

(ii) The following properties are verified:

$$p(p(a_1,c,a_2),d,p(b_1,c,b_2)) = p(p(a_1,d,b_1),c,p(a_2,d,b_2)) \quad (4.2)$$
$$p(0,a,a) = a = p(a,a,1) \quad (4.3)$$
$$p(a,a,0) = 0 \ , \ p(1,a,a) = 1. \quad (4.4)$$

Proof. (i) The proof of Axioms **(G1)** to **(G4)** of a Definition 1 is straightforward:

$$\begin{aligned} p(0,a,1) &= (\overline{a}\wedge 0)\vee(a\wedge 1) = 0\vee a = a \\ p(a,0,b) &= (1\wedge a)\vee(0\wedge b) = a\vee 0 = a \\ p(a,1,b) &= (0\wedge a)\vee(1\wedge b) = 0\vee b = b \\ p(a,b,a) &= (\overline{b}\wedge a)\vee(b\wedge a) = (\overline{b}\vee b)\wedge a = 1\wedge a = a. \end{aligned}$$

For the proof of **(G5)** we have, on the one hand, that

$$\begin{aligned} & p(a,p(x,y,z),b) \\ = & \left(\overline{[(\overline{y}\wedge x)\vee(y\wedge z)]}\wedge a\right)\vee\left([(\overline{y}\wedge x)\vee(y\wedge z)]\wedge b\right) \\ = & \left((\overline{z}\vee\overline{y})\wedge(\overline{x}\vee y)\wedge a\right)\vee(\overline{y}\wedge x\wedge b)\vee(y\wedge z\wedge b) \\ = & (\overline{z}\wedge y\wedge a)\vee(\overline{y}\wedge\overline{x}\wedge a)\vee(\overline{y}\wedge x\wedge b)\vee(y\wedge z\wedge b), \end{aligned}$$

where the following equality has been used

$$(\overline{z}\wedge\overline{x})\vee(\overline{z}\wedge y)\vee(\overline{y}\wedge\overline{x}) = (\overline{z}\wedge y)\vee(\overline{y}\wedge\overline{x}).$$

On the other hand, we have the same result:

$$\begin{aligned} & p(p(a,x,b),y,p(a,z,b)) \\ = & \left(\overline{y}\wedge[(\overline{x}\wedge a)\vee(x\wedge b)]\right)\vee\left(y\wedge[(\overline{z}\wedge a)\vee(z\wedge b)]\right) \\ = & (\overline{y}\wedge\overline{x}\wedge a)\vee(\overline{y}\wedge x\wedge b)\vee(\overline{z}\wedge y\wedge a)\vee(y\wedge z\wedge b). \end{aligned}$$

(ii) To prove (4.2) we begin with the left-hand side of the equality

$$\begin{aligned} & p(p(a_1,c,a_2),d,p(b_1,c,b_2)) \\ = & \left[\overline{d}\wedge((\overline{c}\wedge a_1)\vee(c\wedge a_2))\right]\vee[d\wedge((\overline{c}\wedge b_1)\vee(c\wedge b_2))] \\ = & (\overline{d}\wedge\overline{c}\wedge a_1)\vee(\overline{d}\wedge c\wedge a_2)\vee(d\wedge\overline{c}\wedge b_1)\vee(d\wedge c\wedge b_2), \end{aligned}$$

and show that the right-hand side is equal:

$$\begin{aligned} & p(p(a_1,d,b_1),c,p(a_2,d,b_2)) \\ = & \left[\overline{c}\wedge((\overline{d}\wedge a_1)\vee(d\wedge b_1))\right]\vee\left[c\wedge((\overline{d}\wedge a_2)\vee(d\wedge b_2))\right] \\ = & (\overline{c}\wedge\overline{d}\wedge a_1)\vee(\overline{c}\wedge d\wedge b_1)\vee(c\wedge\overline{d}\wedge a_2)\vee(c\wedge d\wedge b_2). \end{aligned}$$

Property (4.3) corresponds to the idempotency of $\wedge$ and $\vee$ and (4.4) transcribes the definition of complements. □

Lemma 1. *In a system $(A,p,0,1)$ verifying Definition 1, when (4.2) is true then the binary operations $\cdot$ and $\circ$, as in Definition 2, are commutative.*

Proof. Indeed, using **(G4)**, **(G1)** and (4.2),

$$\begin{aligned} a\cdot b = p(0,a,b) &= p(p(0,b,0),a,p(0,b,1)) \\ &= p(p(0,a,0),b,p(0,a,1)) \\ &= p(0,b,a) = b\cdot a. \end{aligned}$$

Similarly, **(G4)**, **(G1)** and (4.2) imply $a\circ b = b\circ a$. □

Proposition 6. *Let $(A,p,0,1)$ be a system verifying Definition 1 such that*

$$p(p(a_1,c,a_2),d,p(b_1,c,b_2)) = p(p(a_1,d,b_1),c,p(a_2,d,b_2)) \quad (4.5)$$
$$p(0,a,a) = a \quad (4.6)$$
$$p(a,a,0) = 0. \quad (4.7)$$

Considering the operations

$$\overline{b} = p(1,b,0),\, b\wedge c = p(0,b,c) \text{ and } a\vee b = p(a,b,1), \quad (4.8)$$

$(A,\wedge,\vee,\overline{()},0,1)$ is a Boolean algebra.

Proof. First, note that, by applying property (2.8), hypothesis (4.6) and (4.7) may be written

$$\begin{aligned} p(a,a,1) &= a; & (4.9) \\ p(1,a,a) &= 1. & (4.10) \end{aligned}$$

Identities are provided by Axioms **(G1)** to **(G5)** as shown in Propositions 2 and 3. Commutativity of $\wedge$ and $\vee$ is guaranteed by Lemma 1. Hypothesis (4.7) and (4.10) give the complements:

$$\begin{aligned} p(a,a,0) = 0 &\overset{(2.6)}{\Rightarrow} p(0,\overline{a},a) = 0 \Rightarrow \overline{a}\wedge a = 0 \\ p(1,a,a) = 1 &\overset{(2.6)}{\Rightarrow} p(a,\overline{a},1) = 1 \Rightarrow a\vee\overline{a} = 1. \end{aligned}$$

In the context of Definition 1, distributivity is a consequence of commutativity and idempotency. First, we obtain the absorption rules:

$$\begin{aligned} a\vee(b\wedge a) &= p(a,p(0,b,a),1) \\ &\overset{(G5)}{=} p(p(a,0,1),b,p(a,a,1)) \\ &\overset{(G1)}{=} p(a,b,p(a,a,1)) \\ &\overset{(4.9)}{=} p(a,b,a) \\ &\overset{(G4)}{=} a, \end{aligned}$$

and, by commutativity of $\vee$, we also get

$$(b\wedge a)\vee a = a \quad (4.11)$$

Then, we have:

$$\begin{aligned} (b\wedge a)\vee(c\wedge a) &= p(p(0,b,a),p(0,c,a),1) \\ &\overset{(G5)}{=} p(p(p(0,b,a),0,1),c,p(p(0,b,a),a,1)) \\ &\overset{(G1)}{=} p(p(0,b,a),c,p(p(0,b,a),a,1)) \\ &= p(p(0,b,a),c,(b\wedge a)\vee a) \\ &\overset{(4.11)}{=} p(p(0,b,a),c,a) \\ &\overset{(G3)}{=} p(p(0,b,a),c,p(0,1,a)) \\ &\overset{(G5)}{=} p(0,p(b,c,1),a) \\ &= (b\vee c)\wedge a. \end{aligned}$$

Similarly, we get

$$(a\vee b)\wedge a = a \text{ and } (a\vee b)\wedge(a\vee c) = a\vee(b\wedge c).$$

□

Proposition 7. *Let $(A, \wedge, \vee, \overline{()}, 0, 1)$ be a Boolean algebra and consider the ternary operation p defined in (4.1). Then, the Boolean algebra obtained from a system $(A, p, 0, 1)$ verifying Definition 1 through Proposition 6 is the same as the initial one.*

Proof. Let us call $(A, \wedge', \vee', ()', 0, 1)$ the Boolean algebra obtained from $(A, p, 0, 1)$. We have:

$$\begin{aligned}
a \wedge' b &= p(0, a, b) = (\overline{a} \wedge 0) \vee (a \wedge b) = 0 \vee (a \wedge b) \\
&= a \wedge b \\
a \vee' b &= p(a, b, 1) = (\overline{b} \wedge a) \vee (b \wedge 1) = (\overline{b} \wedge a) \vee b \\
&= (\overline{b} \vee b) \wedge (a \vee b) = 1 \wedge (a \vee b) = a \vee b \\
a' &= p(1, a, 0) = (\overline{a} \wedge 1) \vee (a \wedge 0) = \overline{a} \vee 0 = \overline{a}.
\end{aligned}$$

□

Proposition 8. *Let $(A, p, 0, 1)$ be an algebra verifying Definition 1 and (4.5) to (4.7), and let $\wedge$, $\vee$ and $\overline{()}$ be the operations defined in (4.8). Then, the algebra obtained from the Boolean algebra $(A, \wedge, \vee, \overline{()}, 0, 1)$ through Proposition 5 is the same as the initial one.*

Proof. Let us call $(A, p', 0, 1)$ the algebra obtained from $(A, \wedge, \vee, \overline{()}, 0, 1)$. We have:

$$\begin{aligned}
p'(a, b, c) &= (\overline{b} \wedge a) \vee (b \wedge c) \\
&= p(p(0, \overline{b}, a), p(0, b, c), 1) \\
&= p(p(a, b, 0), p(0, c, b), 1) \\
&= p(p(a, b, 0), c, p(p(0, a, \overline{b}), b, p(1, a, 1))) \\
&= p(p(a, b, 0), c, p(p(0, b, 1), a, p(\overline{b}, b, 1))) \\
&= p(p(a, b, 0), c, p(b, a, 1)) \\
&= p(p(a, b, 0), c, p(a, b, 1)) \\
&= p(p(a, c, a), b, p(0, c, 1) \\
&= p(a, b, c).
\end{aligned}$$

□

As examples, we present here the ternary operations verifying Axioms **(G1)**—**(G5)** that produce, through Proposition 6, the usual Boolean algebras with 2 and 4 elements. There is, in fact, only such operation with 2 elements which is

p(-,0,-)	0	1
0	0	0
1	1	1

p(-,1,-)	0	1
0	0	1
1	0	1

This gives the usual Boolean algebra:

p(0,-,-)	0	1
0	0	0
1	0	1

p(-,-,1)	0	1
0	0	1
1	1	1

-	p(1,-,0)
0	1
1	0

The only 4-elements ternary operation that verify the hypothesis of Proposition 6 is

p(-,0,-)	0	A	B	1
0	0	0	0	0
A	A	A	A	A
B	B	B	B	B
1	1	1	1	1

p(-,1,-)	0	A	B	1
0	0	A	B	1
A	0	A	B	1
B	0	A	B	1
1	0	A	B	1

.

p(-,A,-)	0	A	B	1
0	0	A	0	A
A	0	A	0	A
B	B	1	B	1
1	B	1	B	1

p(-,B,-)	0	A	B	1
0	0	0	B	B
A	A	A	1	1
B	0	0	B	B
1	A	A	1	1

producing the Boolean algebra with

p(0,-,-)	0	A	B	1
0	0	0	0	0
A	0	A	0	A
B	0	0	B	B
1	0	A	B	1

p(-,-,1)	0	A	B	1
0	0	A	B	1
A	A	A	1	1
B	B	1	B	1
1	1	1	1	1

and

-	p(1,-,0)
0	1
A	B
B	A
1	1

.

5 Boolean algebras from a completely commutative ternary operation

Distributive lattices, and in particular Boolean algebras, were expressed in terms of ternary operations in earlier works ([8], [4], [3], [1], [6]). However, there the approach was different because lattices are obtained from a completely commutative ternary operation, i.e an operation in which each pair of elements may be interchanged without changing its value. But, in a non trivial algebra verifying Definition 1, there is no completely commutative operation as, for instance, **(G2)** would imply

$$a = p(a, 0, b) = p(b, 0, a) = b.$$

Another difference is that, in those earlier works, the involution that originates complements is part of the structure and not defined from the ternary structure. It is comprehensible because, if p is completely commutative **(G1)** implies

$$p(1, a, 0) = p(0, a, 1) = a.$$

The ternary operation used in [4] and [3] is

$$p(a, b, c) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a). \qquad (5.1)$$

For the 2-elements case this operation is:

p(-,0,-)	0	1
0	0	0
1	0	1

p(-,1,-)	0	1
0	0	1
1	1	1

The structure proposed in [4] is $(A, p, ()')$ with

$$\begin{aligned} p(a,b,p(c,d,e)) &= p(p(a,b,c),d,p(a,b,e)) \\ p(a,b,b) &= p(b,b,a) = b \\ p(a,b,b') &= p(b',b,a) = a \end{aligned}$$

and implies complete commutativity of p. Considering two elements $0, 1 \in A$, complements of each other ($0' = 1, 1' = 0$), this structure verify **(G1)**, **(G4)**, **(G5)** and the following instances of **(G2)** and **(G3)**:

$$p(a,0,0) = a;\ p(a,0,1) = a;\ p(0,1,b) = b;\ p(1,1,b) = b.$$

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Appendix A

We present here all the systems $(S, p, 0, 1)$ with a set of four distinct elements $S = \{0, A, B, 1\}$ that verify Definition 1. Properties (2.4) and (2.5) leave two possibilities for the complements:

I) $\overline{A} = B$, or

II) $\overline{A} = A$ and $\overline{B} = B$.

Axioms **(G1)** to **(G3)** imply the following form for the ternary operation. In both cases:

p(0,-,-)	0	A	B	1
0	0	0	0	0
A	0	α_1	α_2	A
B	0	α_3	α_4	B
1	0	A	B	1

p(-,-,1)	0	A	B	1
0	0	A	B	1
A	A	β_1	β_2	1
B	B	β_3	β_4	1
1	1	1	1	1

and

p(-,0,-)	0	A	B	1
0	0	0	0	0
A	A	A	A	A
B	B	B	B	B
1	1	1	1	1

p(-,1,-)	0	A	B	1
0	0	A	B	1
A	0	A	B	1
B	0	A	B	1
1	0	A	B	1

.

For case I), we also have

p(-,A,-)	0	A	B	1
0	0	α_1	α_2	A
A	α_3	A	λ	β_1
B	α_4	μ	B	β_3
1	B	β_2	β_4	1

p(-,B,-)	0	A	B	1
0	0	α_3	α_4	B
A	α_1	A	μ	β_2
B	α_2	λ	B	β_4
1	A	β_1	β_3	1

and for case II), we have

p(-,A,-)	0	A	B	1
0	0	α_1	α_2	A
A	α_1	A	λ	β_1
B	α_2	λ	B	β_3
1	A	β_1	β_3	1

p(-,B,-)	0	A	B	1
0	0	α_3	α_4	B
A	α_3	A	μ	β_2
B	α_4	μ	B	β_4
1	B	β_2	β_4	1

.

Axiom **(G5)** restricts the possible values of the otherwise arbitrary 10 constants α_i, β_i, $i = 0, \ldots, 4$, and λ, μ. The result is that, in an algebra with four elements verifying Definition 1, there are 12 ternary operations verifying I) and 5 verifying II). Let us organize them using other properties they verify.

1. Idempotency and commutativity.
   If $\cdot$ and $\circ$ are idempotent operations then $\alpha_1 = A = \beta_1$ and $\alpha_4 = B = \beta_4$; If they are commutative then $\alpha_2 = \alpha_3$ and $\beta_2 = \beta_3$. In case I) below, (1) is a Boolean algebra and (2) and (3) are distributive lattices. In case II), (4) is a distributive lattice with the same binary operations as the Boolean algebra (1).

   - Case I) $\overline{A} = B, \overline{B} = A$

	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	λ	μ
(1)	A	0	0	B	A	1	1	B	0	1
(2)	A	B	B	B	A	A	A	B	B	A
(3)	A	A	A	B	A	B	B	B	A	B

   - Case II) $\overline{A} = A, \overline{B} = B$

	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	λ	μ
(4)	A	0	0	B	A	1	1	B	A	B

2. Idempotency without commutativity.
   There exist 2 ternary operation in case II) with the same $\cdot$ and $\circ$ operations. Remark: this means that$\{\overline{()}, \cdot, \circ\}$ doesn't determine the ternary operation p in systems verifying Definition 1.

- Case II) $\overline{A} = A, \overline{B} = B$

	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	λ	μ
(5)	A	B	A	B	A	A	B	B	A	A
(6)	A	B	A	B	A	A	B	B	B	B

3. Commutativity without idempotency.

- Case I) $\overline{A} = B, \overline{B} = A$

	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	λ	μ
(7)	1	B	B	0	1	A	A	0	0	1
(8)	0	A	A	1	0	B	B	1	0	1
(9)	B	1	1	A	B	0	0	A	0	1
(10)	A	B	B	0	1	A	A	B	B	A
(11)	0	A	A	B	A	B	B	1	A	B

- Case II) $\overline{A} = A, \overline{B} = B$

	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	λ	μ
(12)	A	A	A	A	A	A	A	A	A	A
(13)	B	B	B	B	B	B	B	B	B	B

4. No idempotency nor commutativity.
(14) and (15) have the same binary operations; (16) and (17) have the same binary operations.

- Case I) $\overline{A} = B, \overline{B} = A$

	α_1	α_2	α_3	α_4	β_1	β_2	β_3	β_4	λ	μ
(14)	0	0	A	B	A	1	B	1	0	1
(15)	0	0	A	B	A	1	B	1	A	B
(16)	A	B	0	0	1	A	1	B	0	1
(17)	A	B	0	0	1	A	1	B	B	A

APPENDIX B

We present here the list of 2-ary and 1-ary operations, constructed from one unique ternary operation p verifying Definition 1. By one unique p, we mean that we are not considering, for instance, the binary operation $p(a, b, \overline{a}) = p(a, b, p(1, a, 0))$ that involves two p.

We have two pairs of dual binary operation that we note $(\wedge, \vee)$ and $(\wedge', \vee')$:

$$\begin{array}{lll} p(0,a,b) = a \wedge b & p(a,b,1) = a \vee b & p(a,0,b) = a \\ p(a,b,0) = \overline{b} \wedge a & p(1,a,b) = b \vee \overline{a} & p(a,1,b) = b \\ p(a,a,b) = a \wedge' b & p(a,b,b) = a \vee' b & p(a,b,a) = a. \end{array}$$

Considering the existence of the binary operations $\wedge$ and $\vee$, only one unary operation is left undefined and that is the complement operation we called $\overline{()}$:

$$\begin{array}{lll} p(a,0,0) = a & p(0,a,0) = 0 & p(0,0,a) = 0 \\ p(a,0,1) = a & p(0,a,1) = a & p(0,1,a) = a \\ p(a,1,0) = 0 & p(1,a,0) = \overline{a} & p(1,0,a) = 1 \\ p(a,1,1) = 1 & p(1,a,1) = 1 & p(1,1,a) = a \end{array}$$

$$\begin{array}{lll} p(0,a,a) = a \wedge a & p(a,a,1) = a \vee a & p(a,0,a) = a \\ p(a,a,0) = \overline{a} \wedge a & p(1,a,a) = \overline{a} \vee a & p(a,1,a) = a \\ & & p(a,a,a) = a. \end{array}$$

Note that the 0-ary operations are all fixed by Axioms **(G2)** and **(G3)**:

$$\begin{array}{lllllll} p(0,0,0) & = & 0 & & p(1,0,0) & = & 1 \\ p(0,0,1) & = & 0 & & p(1,0,1) & = & 1 \\ p(0,1,0) & = & 0 & & p(1,1,0) & = & 0 \\ p(0,1,1) & = & 1 & & p(1,1,1) & = & 1. \end{array}$$

Due to Axiom **(G4)**, the operations $\wedge'$ and $\vee'$ are idempotent. They verify the Morgan relations as a direct consequence of (2.8), but contrarily to $\wedge$ and $\vee$ they do not produce monoids in general within an algebra verifying Definition 1. However, the next proposition shows an interesting interplay between these operations and $\wedge$ and $\vee$.

Proposition 9. *Let $(A, p, 1, 0)$ be a system verifying Definition 1 and the binary operations $\wedge, \vee, \wedge', \vee'$ defined from p as above. If $\wedge'$ is commutative, i.e. $p(a, a, b) = p(b, b, a)$ then:*

$$a \vee a = a, \quad \overline{a} \vee a = 1, \quad a \wedge a = a \quad \textit{and} \quad \overline{a} \wedge a = 0.$$

Proof.

$$a \vee a = p(a, a, 1) = p(1, 1, a) = a$$

$$\overline{a} \vee a = p(\overline{a}, a, 1) = p(1, \overline{a}, \overline{a}) = p(\overline{a}, 1, 1) = 1.$$

If $\wedge'$ is commutative then so is $\vee'$, because $a \vee' b = \overline{\overline{b} \wedge' \overline{a}}$. Explicitly, $p(a, b, b) = p(b, a, a)$ implies

$$a \wedge a = p(0, a, a) = p(a, 0, 0) = a$$

$$\overline{a} \wedge a = p(0, \overline{a}, a) = p(a, a, 0) = p(0, 0, a) = 0.$$

□

APPENDIX C

We define here two other binary operations and present some of their properties.

Definition 4. *Let $(A, p, 0, 1)$ be an algebra such as in Definition 1. The symbols $*$ and $\dagger$ will denote the following binary operations:*

$$a * b = p(\overline{b}, a, b) \quad (5.2)$$

$$a \dagger b = p(a, b, \overline{a}). \quad (5.3)$$

In the canonical example of the unit interval $A = [0, 1]$, where $p(a, b, c) = a - ba + bc$ we have

$$a * b = ab + \overline{a}\overline{b} = 1 - a - b + 2ab$$

$$a \dagger b = a\overline{b} + b\overline{a} = a + b - 2ab.$$

From (2.4), (2.5) and (2.8), we can deduce the following Morgan relations:

$$\overline{a * b} = \overline{b} \dagger \overline{a} \quad (5.4)$$

$$\overline{a \dagger b} = \overline{b} * \overline{a}. \quad (5.5)$$

The next proposition shows another particularity of $*$ and $\dagger$ when related to the unary operation $\overline{()}$.

Proposition 10. *Let $(A, p, 1, 0)$ be as in Definition 1. The following properties hold:*

$$a * b = \overline{a} * \overline{b} \tag{5.6}$$
$$a \dagger b = \overline{a} \dagger \overline{b}. \tag{5.7}$$

Demonstração. This follows immediately from (2.5) and (2.6). □

Of course, we also have:

$$a * b = \overline{b} \dagger a \qquad a \dagger b = b * \overline{a}.$$

Monoids are associated.

Proposition 11. *$(A, *, 1)$ is a monoid:*

$$a * (b * c) = (a * b) * c \tag{5.8}$$
$$a * 1 = a = 1 * a. \tag{5.9}$$

Demonstração. Associativity is proven using **(G5)**:

$$(a * b) * c = p(\overline{c}, p(\overline{b}, a, b), c) = p(p(\overline{c}, \overline{b}, c), a, p(\overline{c}, b, c)) = a * (b * c).$$

Right identity is just **(G1)** and left identity is **(G3)**. □

Proposition 12. *$(A, \dagger, 0)$ is a monoid:*

$$a \dagger (b \dagger c) = (a \dagger b) \dagger c \tag{5.10}$$
$$a \dagger 0 = a = 0 \dagger a. \tag{5.11}$$

Demonstração. Associativity is proven using **(G5)**:

$$(a \dagger b) \dagger c = p(p(a, b, \overline{a}), c, p(a, \overline{b}, \overline{a})) = p(a, p(b, c, \overline{b}), \overline{a}) = a \dagger (b \dagger c).$$

Left identity is just **(G1)** and right identity is **(G2)**. □

Using (2.4), **(G2)** and **(G3)**, we also get directly:

$$a * 0 = \overline{a} = 0 * a \qquad a \dagger 1 = \overline{a} = 1 \dagger a \tag{5.12}$$

The binary operations $*$ and $\dagger$ don't distribute in general but have relations similar to distributivity as shown in the next proposition.

Proposition 13. *Let $(A, p, 1, 0)$ be a supramobi algebra. The following properties hold:*

$$a * (b \dagger c) = b \dagger (a * c) \tag{5.13}$$
$$(a \dagger b) * c = (a * c) \dagger b \tag{5.14}$$
$$a \dagger (b * c) = b * (a \dagger c) \tag{5.15}$$
$$(a * b) \dagger c = (a \dagger c) * b \tag{5.16}$$

Demonstração. The proofs use (2.6), (2.7) and **(G5)**.

$$\begin{aligned} a * (b \dagger c) &= p(p(b, \overline{c}, \overline{b}), a, p(b, c, \overline{b})) \\ &= p(b, p(\overline{c}, a, c), \overline{b}) \\ &= p(b, a * c, \overline{b}) \\ &= b \dagger (a * c) \end{aligned}$$

$$\begin{aligned} (a \dagger b) * c &= p(\overline{c}, p(a, b, \overline{a}), c) \\ &= p(p(\overline{c}, a, c), b, p(\overline{c}, \overline{a}, c)) \\ &= p(a * c, b, \overline{a * c}) \\ &= (a * c) \dagger b \end{aligned}$$

$$\begin{aligned} a \dagger (b * c) &= p(a, p(\overline{c}, b, c), \overline{a}) \\ &= p(p(a, \overline{c}, \overline{a}), b, p(a, c, \overline{a})) \\ &= p(\overline{a \dagger c}, b, a \dagger c) \\ &= b * (a \dagger c) \end{aligned}$$

$$\begin{aligned} (a * b) \dagger c &= p(p(\overline{b}, a, b), c, p(\overline{b}, \overline{a}, b)) \\ &= p(\overline{b}, p(a, c, \overline{a}), b) \\ &= p(\overline{b}, a \dagger c, b) \\ &= (a \dagger c) * b \end{aligned}$$

□

In the context of mobi algebras, a consequence of Proposition 4 related to the operations (5.2) and (5.3) is that they both have ½ as an absorbing element.

Proposition 14. *Let $(A, p, 1, \frac{1}{2}, 0)$ be a mobi algebra. Then, for all $a \in A$, the following properties hold:*

$$a * \tfrac{1}{2} = \tfrac{1}{2} = \tfrac{1}{2} * a \tag{5.17}$$
$$a \dagger \tfrac{1}{2} = \tfrac{1}{2} = \tfrac{1}{2} \dagger a. \tag{5.18}$$

Demonstração. From (3.1) and **(A3)**, we get:

$$a * \tfrac{1}{2} = p(\tfrac{1}{2}, a, \tfrac{1}{2}) = \tfrac{1}{2}, \qquad \tfrac{1}{2} \dagger a = p(\tfrac{1}{2}, a, \tfrac{1}{2}) = \tfrac{1}{2}.$$

From (3.2) and (3.3), we get:

$$\tfrac{1}{2} * a = p(\overline{a}, \tfrac{1}{2}, a) = \tfrac{1}{2}, \qquad a \dagger \tfrac{1}{2} = p(a, \tfrac{1}{2}, \overline{a}) = \tfrac{1}{2}.$$

□

In the context of Boolean algebras, the operation †, defined in(5.3), is just the usual Boolean addition.

A first note on semibiproducts of semigroups

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Abstract The classical notion of biproduct introduced by Mac Lane for additive categories is considered here in the category of semigroups and semigroup homomorphisms. It is called semi-biproduct because two of the four arrows in the classical diagram of biproduct may fail to be homomorphisms. In the particular case of groups it is a generalization of semi-direct product and corresponds precisely to group extensions (not necessarily split). Long ago, Schreier extensions were introduced in monoids to mimic the behaviour of group extensions in groups. As shown recently, semi-biproducts of monoids generalize Schreier extensions and correspond to a certain kind of pseudo-actions with a factor system (as observed in groups) together with a new ingredient which is invisible in groups. This new ingredient was called a correction system and it is responsible for the strange phenomenon that occurs in monoids but not in groups: the middle object in an extension is not bijective as a set to the cartesian product of its two ends. These notions are now transported into the context of semigroups and it is shown that similar results are obtained even though null objects are not present.

keywords Semi-biproduct, biproduct, semi-direct product, groups, monoids, semigroups, pseudo-action, correction factor, factor system, factor set, extension, perturbation system

1 Introduction

The categorical concept of biproduct ([9], p.194), defined for *Ab*-categories, can be adapted to the category of commutative semigroups as a diagram

$$X \underset{k}{\overset{q}{\leftrightarrows}} A \underset{s}{\overset{p}{\rightleftarrows}} B \tag{1.1}$$

in which X, A and B are commutative semigroups, k, s, p and q are semigroup homomorphisms and the following conditions hold

$$kq + sp = 1_A \tag{1.2}$$
$$ps = 1_B \tag{1.3}$$
$$qk = 1_X. \tag{1.4}$$

In an *Ab*-category, such diagram is simultaneously a product and a coproduct. In particular, there exists a null object and the identities $pk = 0$ and $qs = 0$ are derived [9]. Clearly, this is not something that can be expected in the category of commutative semigroups. The same phenomenon is observed in the category of commutative monoids. Contrary to the case of semigroups, in monoids the two extra conditions $pk = 0$ and $qs = 0$ can be added as part of the definition [13]. This is not possible in commutative semigroups due to the lack of a null object. Surprisingly, as we will see, there is a way to work out the notion of biproduct of commutative semigroups. Even more surprisingly, in the fashion of the paper [13], we will see that it is possible to consider a notion of semi-biproduct of semigroups, not necessarily commutative.

Moving to a non-commutative context another difficulty is encountered. Namely that the sum of two homomorphisms may no longer be a homomorphism. This difficulty is well understood in the category of groups and group homomorphisms, where semi-direct products are considered instead. A semi-direct product of groups can be seen as a diagram

$$X \underset{k}{\overset{q}{\leftrightarrows}} A \underset{s}{\overset{p}{\rightleftarrows}} B \tag{1.5}$$

in which X, A and B are groups (not necessarily abelian groups), k, s, p are group homomorphisms while q is a set theoretical map, moreover, conditions (1.2)–(1.4) are satisfied.

As it is well known, the group A is isomorphic to a group $X \rtimes_{\varphi} B$, called the semi-direct product of X and B via the action

$$\varphi\colon B \to \mathrm{Aut}(X),$$

obtained as $\varphi(b)(x) = b \cdot x = q(s(b) + k(x))$, whose group operation is

$$(x, b) + (x', b') = (x + b \cdot x', b + b'). \tag{1.6}$$

Our aim is to study the notion of semi-direct product in the case of semigroups while extending it into the direction of a bi-product, hence the name semi-biproduct.

In this paper, a semi-biproduct of semigroups is a diagram

$$X \underset{k}{\overset{q}{\leftrightarrows}} A \underset{s}{\overset{p}{\rightleftarrows}} B \tag{1.7}$$

in which X, A and B are semigroups (not necessarily commutative), k and p are semigroup homomorphisms while

s and q are set theoretical maps, moreover, the conditions (1.2)–(1.4) are satisfied.

When X, A and B are groups then the deviation ofered by s to be a homomorphism is well understood (see e.g. [10] and its references to previous work). In that case the group A is isomorphic to a group $X \rtimes_{\varphi,\gamma} B$, called the semi-direct product of X and B via the pseudo-action

$$\varphi(b,x) = b \cdot x = q(s(b)+k(x)) = s(b)+k(x)-s(b)$$

and the factor system $\gamma\colon B \times B \to B$

$$\gamma(b,b') = b \times b' = q(s(b)+s(b')) = s(b)+s(b')-s(b+b'),$$

whose group operation is given by the formula

$$(x,b)+(x',b') = (x+(b\cdot x')+(b\times b'), b+b'). \quad (1.8)$$

The case when X, A and B are monoids, k and p are monoid homomorphisms and the set theoretical maps s and q preserve the neutral element is quite different from the case of groups (see [13]). Firstly, the extra conditions $pk = 0$ and $qs = 0$ have to be imposed. Secondly, the monoid A is no longer isomorphic to a monoid $X \rtimes_{\varphi,\gamma} B$ whose underlying set is the cartesian product $X \times B$. As proved in [13], in the case of monoids, there is a new ingredient which is invisible in groups. This new ingredient is called a *correction system* in [13] and it consists of a map $\rho\colon X \times B \to X$ denoted by $\rho(x,b) = x^b$ and obtained as

$$\rho(x,b) = q(k(x)+s(b)).$$

It can be proved [13] that the correction system is trivial, i.e. $\rho(x,b) = x$, as soon as the monoid X admits cancellation on the right and B is a group. To our best knowledge, this is perhaps the reason why this notion of correction system has not been considered before.

The correction system ρ must satisfy some conditions together with the factor system γ and the map φ, which is no longer an action — we call it a pre-action. With these three ingredients at hand we are able to recover the monoid A as being isomorphic to a subset of the cartesian product $X \times B$, namely $R \subseteq X \times B$ defined as[1]

$$(x,b) \in R \Leftrightarrow x^b = x$$

with neutral element $(0,0) \in R$ and the operation

$$(x,b)+(x',b') = ((x+(b\cdot x')+(b\times b'))^{b+b'}, b+b'), \quad (1.9)$$

which can be shown to be well defined on R and associative there. We often write R as $R_{\rho,\varphi,\gamma}$ and thus $A \cong R_{\rho,\varphi,\gamma}$. Clearly, when $\rho(x,b) = x$ is the trivial correction system then we have the same result as for groups with the difference that the extension

$$X \xrightarrow{k} A \xrightarrow{p} B,$$

is a Schreier extension (see [1, 14, 13]), rather than an arbitrary extension. Indeed, asking the correction system to be trivial is the same as asking $q(k(x)+s(b)) = x$, which is precisely one of the conditions required in [1]. Furthermore, in groups, the map q is uniquely determined as $q(a) = a - sp(a)$, while in monoids it is uniquely determined provided the extra conditions $q(k(x)+s(b)) = x$ and $pk = 0$ are satisfied. When that is the case, the map q is uniquely determined as the X-component for the inverse map of $\alpha\colon X \times B \to A$, defined as $\alpha(x,b) = k(x)+s(b)$.

The purpose of this paper is to extend the results obtained in [13] from the context of monoids to the context of semigroups.

As we will see, the case of semigroups is even more surprising than the case of monoids when compared with groups. For example, even when the correction system is trivial, in semigroups it is possible that the semigroup A is not isomorphic to a semigroup whose underlying set is the cartesian product $X \times B$. This new phenomenon is explained by the lack of condition $pk = 0$ which creates the existence of a new homomorphism, $h\colon X \to B$, with $h = pk$. This new ingredient is used in redefining the subset $R = R_{h,\rho,\varphi,\gamma} \subseteq X \times B$ via the formula

$$(x,b) \in R \Leftrightarrow \rho(x,b) = x,\ h(x)+b = b. \quad (1.10)$$

A simple example that illustrates the situation is obtained by considering $X = B = (\{0,1\},\cdot)$, the semigroup with the cardinal numbers 0 and 1 and the usual multiplication between them. With all other ingredients being trivial, that is $\rho(x,b) = x$, $\varphi(b,x) = x$ and $\gamma(b,b') = 1$ for all x and b, there are still three different homomorphisms $h\colon X \to B$ to be considered. The two constant maps, $h(x) = 0$, $h(x) = 1$ and the identity map $h(x) = x$. As expected, when $h(x) = 1$ then $R_1 = X \times B$ is the cartesian product semigroup. When h is the identity homomorphism then $R_h = \{(0,0),(1,0),(1,1)\}$ is a subsemigroup of the cartesian product $X \times B$. When $h(x) = 0$ we obtain $R_0 = \{(0,0),(1,0)\}$ which is itself a subsemigroup of $R_h = \{(0,0),(1,0),(1,1)\}$ and isomorphic to X,

$$X \cong R_0 \hookrightarrow R_h \hookrightarrow R_1 = X \times B.$$

There is one more aspect in which the homomorphism $h = pk$ makes an unexpected appearance when compared with the situation in monoids. That is the formula (1.9) has to be modified to become

$$(x,b)+(x',b') = ((x+(b\cdot x')+((b+h(x'))\times b'))^{b+b'}, b+b'). \quad (1.11)$$

We will show that in every semi-biproduct of semigroups such as in (1.7), the semigroup A is always isomorphic to the set $R = R_{h,\rho,\varphi,\gamma}$ defined as in (1.10) with the binary operation (1.11). We will also see that the notion of pseudo-action of semigroups, in the sense of a homomorphism $h\colon X \to B$ together with a correction system $\rho\colon X \times B \to X$, a pre-action $\varphi\colon B \times X \to X$ and

[1] Note that we are using x^b as $\rho(x,b)$ in the same way as it is customary to use $b \cdot x$ as $\varphi(b,x)$ to represent the action.

a factor system $\gamma\colon B \times B \to X$, satisfying an appropriate condition, is by itself not sufficient to define a semi-biproduct. On the other hand, every semi-biproduct of semigroups induces a pseudo-action of semigroups. A counter example is observed when considering the previous constant homomorphism $h(x) = 0$ together with trivial φ, ρ and γ. It does not create a semi-biproduct diagram because the map $\pi_B\colon R_0 \to B$, with $\pi_B(x, b) = b$, is not surjective and hence it cannot have a right inverse.

2 Semi-biproducts of semigroups

In this section we introduce the notion of semi-biproduct in the category of semigroups and give some remarks on how to generalize it into a wider categorical context.

Definition 5. *A semi-biproduct of semigroups is a diagram*

$$X \underset{k}{\overset{q}{\leftrightarrows}} A \underset{s}{\overset{p}{\rightleftarrows}} B \tag{2.1}$$

in which X, A and B are semigroups, k and p are semigroup homomorphisms, s and q are set theoretical maps and the following conditions are satisfied:

$$kq + sp = 1_A \tag{2.2}$$
$$ps = 1_B \tag{2.3}$$
$$qk = 1_X. \tag{2.4}$$

Equation (2.2) has the obvious meaning that $kq(a) + sp(a) = a$ for all $a \in A$. The definition of semi-biproduct can be generalized into the wider context of a category $\mathbb{C}$, equipped with a bifunctor $H\colon \mathbb{C}^{\mathrm{op}} \times \mathbb{C} \to \mathbf{Set}$ and two natural transformations

$$H \times H \xrightarrow{\;+\;} H \xleftarrow{\;\varepsilon\;} \hom_{\mathbb{C}} .$$

When $\mathbb{C}$ is the category of semigroups then $\hom_{\mathbb{C}}(A, B)$ is the set of semigroup homomorphisms from A to B and we take $H(A, B)$ to be the set of all maps from the underlying set of A into the underlying set of B. The natural transformation ε is the inclusion of a homomorphism as a map while the natural transformation $+$ is the usual component-wise addition of maps in semigroups. In this new language, s and q are elements in the sets $H(B, A)$ and $H(A, X)$, respectively. Then the ambiguous condition $ps = 1_B$ has the specific meaning that $H(1_B, p)(s) = \varepsilon(1_B)$. The other conditions are treated in a similar fashion. Further details are to be explored in future work.

Given a semi-biproduct of semigroups, say

$$X \underset{k}{\overset{q}{\leftrightarrows}} A \underset{s}{\overset{p}{\rightleftarrows}} B \tag{2.5}$$

we put $h = pk$ and consider the maps ρ, φ, γ, defined as

$$\rho(x, b) = q(k(x) + s(b)) \tag{2.6}$$
$$\varphi(b, x) = q(s(b) + k(x)) \tag{2.7}$$
$$\gamma(b, b') = q(s(b) + s(b')) \tag{2.8}$$

for all $x \in X$ and $b, b' \in B$.

The following theorem is a collection of results that are obtained by considering a semi-biproduct of semigroups with h, ρ, φ, γ as above.

Theorem 1. *Let be given a semi-biproduct of semigroups such as the one in (2.5). Then:*

1. *$h(x) = h(x) + h(x)$, for all $x \in X$;*
2. *$p(a) = hq(a) + p(a)$, for all $a \in A$;*
3. *$q(a) = \rho(q(a), p(a))$, for all $a \in A$;*
4. *the following equation holds for every $a, a' \in A$*
$$a+a' = k(qa+\varphi(pa, qa')+\gamma(pa+hqa', pa'))+sp(a+a'); \tag{2.9}$$
5. *the map $\langle q, p\rangle\colon A \to X \times B$ is injective;*
6. *the image of the map $\langle q, p\rangle$ is $R \subseteq X \times B$ defined as*
$$R = \{(x, b) \mid \rho(x, b) = x,\ h(x) + b = b\}; \tag{2.10}$$
7. *the map $\alpha\colon R \to A$, defined as $\alpha(x, b) = k(x)+s(b)$, is a bijection;*
8. *the binary operation on the set $X \times B$, defined as*
$$(x, b)+(x', b') = (\rho(x+\varphi(b, x')+\gamma(b+h(x'), b'), b+b'), b+b') \tag{2.11}$$
is well defined on the set $R \subseteq X \times B$;
9. *the set R equipped with the binary operation (2.11) is a semigroup;*
10. *the map $\alpha\colon R \to A$ is an isomorphism of semigroups with inverse $\beta\colon A \to R$, the map defined as $\beta(a) = (q(a), p(a))$;*
11. *in the diagram*
$$\begin{array}{ccccc} X & \underset{k}{\overset{q}{\leftrightarrows}} & A & \underset{s}{\overset{p}{\rightleftarrows}} & B \\ \Big\| & & \beta\downarrow\uparrow\alpha & & \Big\| \\ X & \underset{\iota_X}{\overset{\pi_X}{\leftrightarrows}} & R & \underset{\iota_B}{\overset{\pi_B}{\rightleftarrows}} & B \end{array} \tag{2.12}$$
where $\pi_B(x, b) = b$, $\pi_X(x, b) = x$, $\iota_B(b) = (qs(b), b)$, $\iota_X(x) = (x, h(x))$, the bottom row is a semi-biproduct of semigroups.

Demonstração. We observe:

1. If starting with $k(x)$ and decomposing it as $kqk(x)+spk(x)$, which is the same as $k(x) + sh(x)$, then we get
$$h(x) = p(k(x)) = p(k(x) + sh(x)) = h(x) + h(x).$$

2. For every $a \in A$,

$$\begin{aligned} p(a) &= p(kq(a)+sp(a)) \\ &= pkq(a)+psp(a) \\ &= hq(a)+p(a). \end{aligned}$$

3. For every $a \in A$,

$$q(a) = q(kq(a)+sp(a)) = \rho(q(a),p(a)).$$

4. Replacing φ and γ in the equation we have $a+a'$ as

$$k(qa+q(spa+kqa')+q(s(pa+hqa')+spa'))+sp(a+a')$$

which is obtained as[2]

$$\begin{aligned} a+a' = (kqa+spa)+(kqa'+spa') = \\ kqa+(spa+kqa')+spa' = \\ kqa+(kq(u)+sp(u))+spa' = \\ kqa+kq(u)+(sp(u)+spa') = \\ v+(sp(u)+spa') = \\ v+kq(sp(u)+spa')+sp(sp(u)+spa') = \\ v+kq(s(pa+hqa')+spa')+s(p(u)+pa') = \\ v+kq(s(pa+hqa')+spa')+sp(u+a') = \\ k(qa+q(u)+q(s(pa+hqa')+spa'))+sp(a+a') \end{aligned}$$

with $p(u+a')$ being the same as $p(a+a')$ due to $u = spa+kqa'$ and $p(spa+kqa'+a') = pa+p(kqa'+spa') = pa+pa'$.

5. The map $a \mapsto (q(a),p(a))$ is injective for if $(qa,pa) = (qa',pa')$ then $a = kqa+spa = kqa'+spa' = a'$.

6. If $a \in A$ then $(qa,pa) \in R$; as a matter of fact we have already seen that $\rho(q(a),p(a)) = q(a)$ and $hq(a)+p(a) = p(a)$. Similarly, if $(x,b) \in R$ then there exists $a \in A$, namely $a = kx+sb$, with $q(a) = x$ and $p(a) = b$. Indeed, because $(x,b) \in R$ we have $q(a) = q(kx+sb) = \rho(x,b) = x$ and $p(a) = p(kx+sb) = h(x)+b = b$.

7. On the one hand we have

$$a \mapsto (qa,pa) \mapsto kqa+spa = a$$

while on the other hand

$$\begin{aligned} (x,b) \mapsto kx+sb \mapsto \\ \mapsto (q(kx+sb),p(kx+sb)) = (\rho(x,b),h(x)+b) \end{aligned}$$

and if $(x,b) \in R$ then $(\rho(x,b),h(x)+b) = (x,b)$.

8. Considering $a = kx+sb$ and $a' = kx'+sb'$ we have $(q(a+a'),p(a+a')) \in R$; by the item (4) in the theorem we know that $(q(a+a'),p(a+a'))$ is precisely

$$(\rho(x+\varphi(b,x')+\gamma(b+h(x'),b'),b+b'),b+b')$$

as soon as (x,b) and (x',b') are both in R.

9. Since α is a bijection and the operation in R is obtained as $(x,b)+(x',b') = (q(a+a'),p(a+a'))$ with $a+a' = kx+sb+kx'+sb'$ it follows that it must be associative;

10. and α is an isomorphism with inverse $\beta(a) = (q(a),p(a))$.

11. We have $\pi_B\iota_B = 1_B$ and $\pi_X\iota_X = 1_X$. In order to prove

$$\iota_X\pi_X+\iota_B\pi_B = 1_R$$

first we observe that the identities

$$q\alpha = \pi_X \tag{2.13}$$
$$\beta k = \iota_X \tag{2.14}$$
$$p\alpha = \pi_B \tag{2.15}$$
$$\beta s = \iota_B \tag{2.16}$$

hold true and then we compute

$$1_R = \beta\alpha = \beta(kq+sp)\alpha = \beta kq\alpha+\beta sp\alpha = \iota_X\pi_X+\iota_B\pi_B.$$

This shows that the bottom row in the diagram is a semi-biproduct of semigroups as desired. □

3 Some properties and examples

Let us analyse first the case of groups and compare it with the classical results of Schreier (1926) and Eilenberg-Mac Lane (late 1940-s) which have been further expanded by many authors since then. It is not difficult to see that in groups the well-known correspondence between semi-direct products and split extensions is expanded into a correspondence between semi-biproducts and extensions with a specified section (but the section need not be a homomorphism). This is sometimes called Schreier's theory of nonabelian group extensions [11] and leads to the low dimensional nonabelian group cohomology. The classical way to obtain nonabelian group 2-cocycles from a group extension as above starts with choosing a set-theoretic section of $p : A \to B$, say s and then use it to define a factor set which measures how far the extension is from being a group homomorphism. However, in our case the section map s is already part of the structure.

We will denote a semi-biproduct of semigroups such as the one in (2.5) as a tuple (X,A,B,p,q,k,s) and associate to it the tuple (h,ρ,φ,γ) with $h = pk$ and ρ,φ,γ the maps defined as in (2.6)–(2.8).

[2](let $u = spa+kqa'$ and $v = kqa+kq(u)$ for simplicity of notation)

Proposition 15. *Let $p\colon A \to B$ be a surjective group homomorphism with a specified section map $s\colon B \to A$, i.e., $ps = 1_B$. Then the tuple (X, A, B, p, q, k, s) is a semi-biproduct of groups if $k\colon X \to A$ is the kernel of p and the map $q\colon A \to X$ is such that $kq(a) = a - sp(a)$, for all $a \in A$. Moreover, if the section s is a group homomorphism then the semi-biproduct (X, A, B, p, q, k, s) is a semi-direct product.*

Proposition 16. *Let (X, A, B, p, q, k, s) be a semi-biproduct of semigroups with associated tuple $(h, \rho, \varphi, \gamma)$ such as the one in (2.5). If X, A and B are groups, then:*

1. *the map q is uniquely determined as $kq(a) = a - sp(a)$, for all $a \in A$;*
2. *$h = pk$ is the trivial homomorphism;*
3. *the map ρ is uniquely determined as $\rho(x, b) = x$ for all $(x, b) \in X \times B$;*
4. *k is the kernel of p;*
5. *p is the cokernel of k;*
6. *the maps φ and γ encode the information of a pseudo action with a factor system and A is isomorphic to the group $X \rtimes_{\varphi,\gamma} B$ whose operation is*
$$(x, b) + (x', b') = (x + \varphi(b, x) + \gamma(b, b'), b + b');$$
7. *the maps φ and γ encode the information of a lax-functor*
$$F\colon B \to \mathbf{Grp}$$
with $F(b)\colon X \to X$ as $F(b)(x) = \varphi(b, x)$ and
$$F_{b,b'}\colon F(b + b') \Longrightarrow F(b)F(b')$$
as $F_{b,b'} = \gamma(b, b') \in X$.

Demonstração. We observe:

1. follows from $kq(a) + sp(a) = a$ for all $a \in A$;
2. follows from Theorem 1(1);
3. using the fact that $h = pk = 0$ we have
$$\begin{aligned} kx + sb &= kq(kx + sb) + sp(kx + sb) \\ &= kq(kx + sb) + s(pkx + psb) \\ &= kq(kx + sb) + s(hx + b) \\ &= kq(kx + sb) + s(b) \end{aligned}$$
and cancelling out $s(b)$ on both sides we obtain $k(x) = kq(k(x) + s(b))$ from which we conclude $\rho(x, b) = x$;
4. follows from Corollary 2, see below;
5. follows from Corollary 1, see below;
6. it is a classical result from Schreier theory, see also the discussion around equations (4.5)–(4.7);
7. it is a classical result which makes use of the so called Grothendieck construction which considers B as a one object groupoid and the functor F takes values in **Grp**, the category of all groups, and sends the one object in the groupoid B to the group X, see also the discussion around equations (4.5)–(4.7);

□

Summing up, in the context of groups, from every semi-biproduct (X, A, B, p, q, k, s) we can extract a group extension $X \to A \to B$ with associated maps φ and γ as pseudo-action and factor system, respectively (further detalis are given in the discussion around equations (4.5)–(4.7)). On the other hand, every group extension with a specified section gives rise to a semi-biproduct. However, the same extension

$$X \xrightarrow{k} A \xrightarrow{p} B$$

if considered with different sections may give inequivalent semi-biproducts due to the fact that an isomorphism between semi-biproducts has to be compatible with the maps q and s whereas an isomorphism of extensions only has to be compatible with p and k.

Let us now analyse the general case of semigroups and investigate how to extend the notions of kernel and cokernel not assuming the existence of a null object.

We will proceed as follows. Instead of the null map $pk = 0$ we consider an arbitrary homomorphism $h = pk$ and work towards a notion which reduces to a kernel when $h = 0$.

And we do the same to capture the notion of cokernel. However, there is a lack of duality between the two notions which can be explained by observing that the equality

$$(kq + sp)f = kqf + spf$$

always holds, even when f is a map, whereas in order to ensure that

$$f(kq + sp) = fkq + fsp$$

holds true we should require f to be a homomorphism.

We start by looking at cokernels in semigroups associated with a semi-biproduct diagram.

Proposition 17. *Let $(X.B, A, p, q, k, s)$ be a semi-biproduct of semigroups. For every semigroup homomorphism $f\colon A \to Z$, with $f = fsp$, there exists a unique semigroup homomorphism $\bar{f}\colon B \to Z$ such that $f = \bar{f}p$.*

Demonstração. The map fs is a homomorphism

$$\begin{aligned} fs(b + b') = fs(psb + psb') = fsp(sb + sb') = \\ f(sb + sb') = fs(b) + fs(b') \end{aligned}$$

and so $\bar{f} = fs$ is one solution. To prove uniqueness we observe that if $\bar{f}$ is such that $\bar{f}p = f$ then

$$\bar{f} = \bar{f}ps = fs.$$

□

Corollary 1. *In the category of monoids, if $h = 0$ then p is the cokernel of k.*

Demonstração. If $fk = 0$ then $f = f(kq + sp) = fkq + fsp = fsp$, and using the previous proposition we conclude that there exists a unique $\bar{f}$ such that $\bar{f}p = f$. Moreover, when $pk = 0$, the condition $f = fsp$ is equivalent to $fk = 0$. Indeed, if $f = fsp$ then $fk = fspk = 0$. □

Contrary to the case of groups, in monoids, we can have a semi-biproduct in which $h = pk$ is not the trivial homomorphism. Take A to be any monoid of idempotents, that is, $a = a + a$ for all $a \in A$, then the diagram

$$A \underset{1_A}{\overset{1_A}{\leftrightarrows}} A \underset{1_A}{\overset{1_A}{\rightleftarrows}} A$$

is a semi-biproduct and $h = 1_A$.

Let us now investigate the notion of kernel in semigroups. In this case, even if $h = pk$ is not the trivial homomorphism we observe that the condition $pf = hqf$ implies the condition $f = kqf$.

Proposition 18. *Let $(X.B, A, p, q, k, s)$ be a semi-biproduct of semigroups. For every semigroup homomorphism $f\colon Z \to A$, with $pf = hqf$, there exists a unique semigroup homomorphism $\bar{f}\colon Z \to X$ such that $f = k\bar{f}$.*

Demonstração. Firstly we observe that if $pf = hqf$ then $f = kqf$. Indeed,

$$f = (kq + sp)f = kqf + spf = kqf + shqf$$

and having in mind that h is pk and that $qk = 1_X$ we have

$$f = kqkqf + spkqf = (kq + sp)kqf = kqf.$$

Secondly, we observe that kqf being a homomorphism implies qf being a homomorphism too. Consequently $\bar{f} = qf$ is one solution. To prove uniqueness we observe that if $\bar{f}$ is such that $k\bar{f} = f$ then $k\bar{f} = f = kqf$ which implies $qk\bar{f} = qkqf$ or $\bar{f} = qf$. □

Corollary 2. *In the category of monoids, if $h = 0$ then k is the kernel of p.*

Demonstração. When $h = 0$ the previous proposition asserts precisely that k is the kernel of p. □

We end this section with the list of all semi-biproducts of semigroups with fixed ends of order 2 whose middle object is of order 3.

Let B be a two element set, say $\{1, 2\}$, and consider the four possible semigroup structures on it represented by the following multiplication tables

$$B_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, B_2 = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, B_3 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, B_4 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{3.1}$$

We let $X_i = B_i$ for $i = 1, 2, 3, 4$ and display the number of semi-biproducts (in which A is order 3) with ends X_i and B_j for all possible cases.

	B_1	B_2	B_3	B_4
X_1	2	0	2	0
X_2	4	0	0	0
X_3	2	0	0	0
X_4	0	0	0	0

Here is a detailed list for each case:

1. The two cases with ends X_1 and B_1 have the same maps p, q, k, s, defined as

$a \in A$	$p(a)$	$q(a)$
1	1	1
2	2	2
3	1	2

$x = b$	$k(x)$	$s(b)$
1	1	3
2	2	2

and two different multiplication tables for the middle object in the semi-biproduct

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

2. The two cases with ends X_1 and B_3 have the same maps p, k, s, and the same multiplication table for the middle object A, defined as

$a \in A$	$p(a)$
1	1
2	1
3	2

$x = b$	$k(x)$	$s(b)$
1	1	2
2	2	3

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix}$$

and admit two different maps q_1, q_2 as tabulated

$a \in A$	$q_1(a)$	$q_2(a)$
1	1	1
2	2	2
3	1	2

3. The four cases with ends X_2 and B_1 have the same maps p, k, and the same multiplication table for the middle object A, defined as

$a \in A$	$p(a)$
1	2
2	2
3	1

x	$k(x)$
1	1
2	2

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix}$$

and admit two different possibilities for the map s combined with two different possibilities for the map q as tabulated

$a \in A$	$q_1(a)$	$q_2(a)$
1	1	1
2	2	2
3	1	2

b	$s_1(b)$	$s_2(b)$
1	3	3
2	1	2

4. The two cases with ends X_3 and B_1 have the same maps p, k, s, and the same multiplication table for the middle object A, defined as

$a \in A$	$p(a)$
1	2
2	2
3	1

$x = b$	$k(x)$	$s(b)$
1	1	3
2	2	1

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 3 \end{bmatrix}$$

and admit two different maps q_1, q_2 as tabulated

$a \in A$	$q_1(a)$	$q_2(a)$
1	1	1
2	2	2
3	1	2

In the case of monoids, when $h = 0$ and the map ρ is trivial, that is $\rho(x, b) = x$ then a semi-biproduct is precisely a Schreier extension with a chosen section and most of the results follow as in groups (see refs).

Let us now investigate further the notion of pseudo action as a structure $(h, \rho, \varphi, \gamma)$ associated to a semi-biproduct of semigroups.

4 Pseudo actions of semigroups

Let X and B be semigroups. A pseudo-action of B on X consists of a homomorphism $h\colon X \to B$, called the perturbation homomorphism, together with three more components:

1. a map $\rho\colon X \times B \to X$, called the correction system and sometimes written as $\rho(x, b) = x^b$;
2. a map $\varphi\colon B \times X \to X$, called the pre-action and sometimes written as $\varphi(b, x) = b \cdot x$;
3. a map $\gamma\colon B \times B \to X$, called the factor system and sometimes written as $\gamma(b, b') = b \times b$.

The set $X \times B$ equipped with the binary operation

$$(x, b)+(x', b') = (\rho(x+\varphi(b, x')+\gamma(b+h(x'), b'), b+b'), b+b') \tag{4.1}$$

is denoted $X \rtimes_{h,\rho,\varphi,\gamma} B$, while the subset $R_{h,\rho,\varphi,\gamma} \subseteq X \times B$ is

$$R_{h,\rho,\varphi,\gamma} = \{(x, b) \mid \rho(x, b) = x,\ h(x) + b = b\}. \tag{4.2}$$

The data $(h, \rho, \varphi, \gamma)$ should satisfy appropriate conditions so that in the case when it is obtained from a semi-biproduct such as in equations (2.6)–(2.8) then the semi-biproduct can be recovered. There is clearly one difficulty, namely there is a missing piece of information: the map qs. Nevertheless, as show by the following examples, it is clear that the existence of such a map is a property of the structure $(h, \rho, \varphi, \gamma)$. In order to see it let us consider the following relevant conditions that can be imposed on the structure $(h, \rho, \varphi, \gamma)$ with $R = R_{h,\rho,\varphi,\gamma}$ and $X \rtimes_{h,\rho,\varphi,\gamma} B$ as above:

(C0) $X \rtimes_{h,\rho,\varphi,\gamma} B$ is a semigroup;

(C1) if $(x, b), (x', b') \in R$ then $(x, b) + (x', b') \in R$

(C2) if $a = (x, b), a' = (x', b'), a'' = (x'', b'') \in R$ then

$$a + (a' + a'') = (a + a') + a''$$

(C3) $\rho(x, h(x)) = x$ and $h(x) + h(x) = h(x)$ for all $x \in X$

(C4) for all $x, x' \in X$,

$$\rho(x+\varphi(h(x), x')+\gamma(h(x+x'), h(x')), h(x+x')) = x+x'$$

(C5) for every $(x, b) \in R$, there exists $x_b \in X$ such that $h(x_b) + b = b$, $\rho(x_b, b) = x_b$ and

$$(x, b) = (x, h(x)) + (x_b, b)$$

(C6) for every $b \in R$, there exists $t(b) \in X$ such that $h(t(b)) + b = b$, $\rho(t(b), b) = t(b)$ and for every $x \in X$ if $(x, b) \in R$ then

$$(x, b) = (x, h(x)) + (t(b), b)$$

The magma structure $X \rtimes_{h,\rho,\varphi,\gamma} B$ is a semigroup if and only if the following identity holds for all $x, x', x'' \in X$ and $b, b', b'' \in B$

$$\rho(u + \varphi(b + b', x'') + \gamma(b + b' + h(x''), b''), b + b' + b'') = \rho(x + \varphi(b, u') + \gamma(b + h(u'), b' + b''), b + b' + b'') \tag{4.3}$$

with u and u' as

$$u = \rho(x + \varphi(b, x') + \gamma(b + h(x'), b'), b + b')$$

and

$$u' = \rho(x' + \varphi(b', x'') + \gamma(b' + h(x''), b''), b' + b'')$$

that are used with the purpose of abbreviating the expression. When B is a monoid and if h is the trivial homomorphism then $R = X \rtimes_{h,\rho,\varphi,\gamma} B$ if and only if ρ is trivial, that is, $\rho(x, b) = x$ for all $x \in X$ and $b \in B$. In that case R is a semigroup if and only if

$$u + \varphi(b + b', x'') + \gamma(b + b', b'') = x + \varphi(b, u') + \gamma(b, b' + b'') \tag{4.4}$$

with

$$u = x + \varphi(b, x') + \gamma(b, b')$$
$$u' = x' + \varphi(b', x'') + \gamma(b', b'').$$

Let us for a moment suppose that X is a monoid too and that $\gamma(0, b) = \gamma(b, 0) = 0$, $\varphi(0, x) = x$ and $\varphi(b, 0) = 0$. Moreover:

1. if $x = 0$, $b' = 0$, $b'' = 0$ then

   $$\varphi(b, x' + x'') = \varphi(b, x') + \varphi(b, x'') \tag{4.5}$$

   that is, each $\varphi(b, -) \colon X \to X$ is a monoid homomorphism;

2. if $x = 0$, $x' = 0$, $b'' = 0$ then

   $$\gamma(b, b') + \varphi(b + b', x'') = \varphi(b, \varphi(b', x'')) + \gamma(b, b')$$

   which in the case X is a group is more commonly written as

   $$\varphi(b, \varphi(b', x'')) = \gamma(b, b') + \varphi(b + b', x'') - \gamma(b, b') \tag{4.6}$$

   and it explains why the map γ can be seen as a natural transformation associated to the lax-functor $F(b)(x) = \varphi(b, x)$, when B is considered as a one object category (see also the last two items on Proposition 16);

3. if $x = 0$, $x' = 0$, $x'' = 0$ then

   $$\gamma(b, b') + \gamma(b + b', b'') = \varphi(b, \gamma(b', b'')) + \gamma(b, b' + b'')$$

   which in the case X is a group is commonly written as

   $$b \cdot \gamma(b', b'')) + \gamma(b, b' + b'') - \gamma(b + b', b'') - \gamma(b, b') = 0 \tag{4.7}$$

   and it is also known as the 2-cocycle condition. We remark that neither X nor B have to be commutative.

Conditions (C1) and (C2) ensure that R is a semigroup with the structure inherited from $X \rtimes_{h,\rho,\varphi,\gamma} B$ even thought $X \rtimes_{h,\rho,\varphi,\gamma} B$ may not be a semigroup.

Conditions (C1)–(C4) ensure that the map $\iota_X \colon X \to R$, with $\iota_X(x) = (x, h(x))$ is well defined and is a homomorphism of semigroups.

Condition (C6), under the assumptions of conditions (C1)-(C4), says that there exists a map $t \colon B \to X$ such that the diagram

$$X \underset{\iota_X}{\overset{\pi_X}{\rightleftarrows}} R \underset{\iota_B}{\overset{\pi_B}{\rightleftarrows}} B, \tag{4.8}$$

where $\pi_B(x, b) = b$, $\pi_X(x, b) = x$, $\iota_B(b) = (t(b), b)$, $\iota_X(x) = (x, h(x))$, is a semi-biproduct of semigroups. Condition (C5) is a weaker version of (C6) which does not guarantee the existence of the map t.

As a consequence of Theorem 1, it is clear that every semi-biproduct of semigroups gives rise to a structure $(h, \rho, \varphi, \gamma)$ satisfying conditions (C1)-(C6). However, in order to recover the semi-biproduct out of its associated structure, $(h, \rho, \varphi, \gamma)$, we would need to keep track of the map qs. Otherwise there is no guarantee that condition (C6) would give the right map $t = qs$.

We end this section with a list of examples of structures $(h, \rho, \varphi, \gamma)$ satisfying several combinations of conditions (C0)-(C6). The pursue of a categorical equivalence between semi-biproducts of semigroups and pseudo actions, although desirable, is postponed to a future work.

The following list of examples was generated in order to illustrate the variety of situation that can occur in a structure such as $(h, \rho, \varphi, \gamma)$ concerning all possible combinations of conditions (C0)-(C6) as defined above.

5 Conclusions and future work

We have introduced the notion of semi-biproduct based on the classical one of biproduct by permitting that 2 of its 4 morphisms be maps instead. This idea was explored here in the context of semigroups but it is clear that it makes sense in magmas (unitary or not). As briefly mentioned after Definition 5, an appropriate categorical context that can be used to study such unusual concept as to involve maps and morphisms on the same ground is the one of a category equipped with a set-valued covariant bifunctor H and two natural transformations $+ \colon H \times H \to H$ and $\varepsilon \colon \hom \to H$. Clearly, $\hom(A, B)$ is the set of morphisms from A to B while, intuitively, $H(A, B)$ can be thought of as a set of maps from A to B (but of course it may as well be any other thing). In practice, $H(A, B)$ can be any collection of weaker forms of morphisms, such as continuous maps in the category of topological groups. There, the natural transformation $+$ is component-wise addition of continuous maps while $\varepsilon(f)$ is the inclusion of a continuous homomorphism $f \in \hom(A, B)$ as a continuous map $\varepsilon(f) \in H(A, B)$.

In this more general setting, a semi-biproduct consists of a sequence

$$X \xrightarrow{k} A \xrightarrow{p} B$$

together with $s \in H(B, A)$ and $q \in H(A, X)$ such that $kq + sp = \varepsilon(1_A)$, $ps = \varepsilon(1_B)$ and $qk = \varepsilon(1_X)$. The notation $guf = H(f, g)(u)$ is useful and particularly successful in expressing $kq = H(1, k)(q)$, $sp = H(p, 1)(s)$, $qk = H(k, 1)(q)$ and $ps = H(1, p)(s)$. Further details can be found in [3, 12, 13].

The difficulty that we have encountered in having an equivalence between semi-biproducts and pseudo-actions is now evident. The missing piece of information that is lost in translation while associating a pseudo-action to a semi-biproduct, namely the map qs, may not even be defined in the general context of a bifunctor H. For example we could have H as a very special class of maps which would not be closed under composition. This suggests a deeper study on the general categorical context as to provide an answer to the problem in semigroups.

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